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# Constructions for the Theorem of Bröcker–Scheiderer

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# Contents

# 1 Introduction

The present work deals with some aspects of Real Algebraic Geometry. This recent branch of mathematics can be seen as the real counterpart of Algebraic Geometry and indeed there are many similarities. However, Real Algebra and Real Algebraic Geometry require special methods and techniques. For instance, in Real Algebraic Geometry one uses succesfully model-theoretic arguments, quadratic forms, general valuation rings, whereas these notions are less important in Algebraic Geometry. Some other notions or tools, like *variety*, *blowing up*, *divisors* or *Zariski–Spectrum* carry over to the real case where they have real counterparts.

We want to give some indications, what Real Algebra and Real Algebraic Geometry means. In Real Algebra, one investigates the laws of ordered structures, like ordered fields or ordered rings. It can be seen as the real counterpart of Commutative Algebra. The name is deduced from the best known example of an ordered field, namely $\mathbb{R}$. Stimulated by Hilbert's $17^{\text{th}}$ problem, Artin and Schreier were the first to consider ordered fields more generally. This has lead to the solution of this famous problem. Later, M.-F. Roy and M. Coste have found the real analogy to the Zariski–Spectrum. This discovery was the starting point of many further investigations in Real Algebra. On the other hand, Real Algebraic Geometry is concerned not only with systems of algebraic equations, but with systems of inequalities, where one uses an ordering of the base field. This geometry has a very rich structure, there are many theorems of finiteness, triviality or local structure.

One of the most surprising theorems gives information about the way of describing a certain kind of semialgebraic sets, the so-called *basic open* sets. For each real variety, we have an invariant, called the *stability index*, which contains much information about the variety. There are several characterizations of the stability index, for our purpose we need the following: Every set of the form

$$\{x \in V : f_1(x) > 0, \ldots, f_n(x) > 0\}$$

with $V$ a real variety and $f_1, \ldots, f_n \in R[V]$ can be described in the same way with only $s(V)$ functions $g_1, \ldots, g_{s(V)}$. For exact definitions of these notions the reader is refered to Chapter 2.

In many cases, the stability index can be computed. The first one to give bounds of this invariant was Bröcker, who used tricky pasting techniques. However, it turned out that his bounds were not the best possible. The exact value of the stability index was found independently by Bröcker and Scheiderer in 1989. Meanwhile, various other proofs of this theorem are known, interesting bounds for the degrees of the $g_i$'s have been found and similar questions in semianalytic geometry have been solved. But so far, no elementary and geometric proof is known. Here, we will use elementary means to show some weaker versions of the Theorem of Bröcker-Scheiderer.

This work is divided into four parts, each of them is more or less independent of the others, with the exception of the first chapter, which is underlying to all the others, since basic definitions and tools like *real variety*, the *Tarski–Seidenberg–Priciple* or *semialgebraic set* are introduced. The content of the first chapter is well-known, whereas the results of the other chapters are new (if not otherwise specified).

The second chapter can be considered as an application of Analysis to Real Algebraic Geometry. Based on an idea about Convex Interpolation in [12], we show an interesting and general theorem which could have other applications in different branches of mathematics. Here we point out how to use it for the problem of reducing inequalities. We solve completely the first interesting case of the Theorem of Bröcker-Scheiderer. Although this is naturally a consequence of the general theorem, no direct proof was known. The proof we give here is simple and understandable.

In the third chapter we generalize some of the ideas arising from the proof in the second chapter. Unfortunately, we don't know how to apply the Theorem of Convex Interpolation to higher-dimensional cases. But the structure of the solutions found in the two-dimensional case gives useful hints for the higher-dimensional case. To begin with, we show a generalization of the theorem in the two-dimensional case. Afterwards, with the help of a very important tool (the *value set*), we give a proof of a weak version of the Theorem of Bröcker–Scheiderer. The advantage of our proof is to yield to an algorithm and to be very geometrical.

The last chapter gives some new results about the reduction of semialgebraic sets. We show that in different situations, special kinds of reductions, which we call *polynomial reductions*, can be found. This gives information about the way the functions $g_i$ depend on the functions $f_i$ of the given description of a basic open set. It is shown that one can choose them by starting with the functions $f_i$ and finitely many additions and multiplications. This result is not trivial, even in the one-dimensional case it is not obvious.

# 2 Real Algebra

In this chapter we collect some basic facts from Real Algebra. As all of this can be found in the references [1],[2] and [3], we don't give any proof. We begin with the introduction of the notions of *ordered field*, *real closed field*, *archimedean field* and *real variety*. Afterwards, the central objects of Semialgebraic Geometry, the *semialgebraic sets* and *semialgebraic functions* are defined and the most important properties are listed. In the end of this chapter, we define the *Real Spectrum* of a ring, which is a very strong tool in Real Algebraic Geometry. However, we won't use it very often in the sequel, since our proofs work on the semialgebraic level. It should not be very hard to generalize some of the constructions to the real spectrum, we don't want to do this.

## 2.1 Ordered fields and real closed fields

The theory of ordered fields was developed by Artin and Schreier in the 20's in order to give a solution to Hilbert's 17th problem. Their work is the foundation of modern real algebra.

**Definition 2.1.1** *A* **preordering** *of a field $F$ is a subset $P$ of $F$ satisfying*

*a)* $P + P \subseteq P$ *and* $P * P \subseteq P$

*b)* $P \cap (-P) = \{0\}$

*c)* $F^2 \subseteq P$

**Definition 2.1.2** *An* **ordering** *of a field $F$ is a preordering $P$ such that $P \cup (-P) = F$.*

Let $\sum F^2 = \{a_1^2 + \ldots + a_n^2 : n \in \mathbb{N}, a_1, \ldots, a_n \in F\}$.

**Proposition 2.1.3** *Let $F$ be a field. Then:*

*a)* $\sum F^2$ *is contained in every preordering.*

*b)* $\sum F^2$ *is a preordering if and only if* $-1 \notin \sum F^2$.

*c) Every preordering is contained in an ordering.*

*d) Every preordering $P$ is the intersection of the orderings containing $P$.*

**Definition 2.1.4** *The field $F$ is called* **formally real***, if $-1 \notin \sum F^2$, that is, if it can be ordered. An* **ordered field** *is a formally real field $F$ equipped with an ordering $P$.*

Given an ordered field $(F, P)$, we write $a \leq_P b$ or just $a \leq b$ instead of $b - a \in P$. This defines a total ordering on $F$ which is compatible with $+$ and $\cdot$ (the operations of $F$).

**Examples 2.1.5**

a) $\mathbb{R}$ has a unique ordering, which is given by the squares.

b) $\mathbb{Q}$ has a unique ordering.

c) $\mathbb{Q}(\sqrt{2})$ has exactly two orderings which come from the two embeddings $\mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{R}$.

**Definition 2.1.6** *A* **real closed field** *is a formally real field $F$ which admits no proper algebraic formally real extension.*

**Proposition 2.1.7** *For a field $F$, the following conditions are equivalent:*

*a) $F$ is real closed.*

*b) There is an ordering $P$ on $F$ which cannot be extended to an algebraic extension of $F$.*

*c) $\sum F^2$ is an ordering and any polynomial of odd degree has a root in $F$.*

*d) $F$ is not algebraically closed, but $F(\sqrt{-1})$ is algebraically closed.*

*If this is the case, then $\sum F^2$ is the only ordering of $F$.*

**Examples 2.1.8**

a) $\mathbb{R}$ is a real closed field, since every positive real number has a square root and every polynomial of odd degree admits a root.

b) $\mathbb{Q}$ is not real closed, since for instance $x^3 - 2$ has no root in $\mathbb{Q}$.

**Proposition and Definition 2.1.9** *Let $(F, P)$ be an ordered field with algebraic closure $\overline{F}$. Then there exists a real closed field $R$ with $F \subseteq R \subseteq \overline{F}$ such that the unique ordering of $R$ extends $P$ and any two real closed fields with that property are conjugate in $\overline{F}$ over $F$. The field $R$ is called the* **real closure** *of $(F, P)$.*

**Example 2.1.10** The real closure of $\mathbb{Q}$ is the set of real algebraic numbers, denoted by $\mathbb{R}_{\text{alg}}$.

**Proposition 2.1.11 (Descartes' Rule)** *Let $R$ be a real closed field and let $f(t) = c_0 t^n + c_1 t^{n-1} + \ldots + c_n$ be a real polynomial with $n \geq 1$ and $c_0 \neq 0$. If $c_n \neq 0$ and all the roots of $f$ are real, then the number of positive roots of $f$ equals the number of variations of signs in the sequence $c_0, \ldots, c_n$ and the number of negative roots of $f$ equals the number of sign changes in the sequence $c_0, -c_1, \ldots, (-1)^n c_n$.*

See [1] for a proof. It is surprising, but this Proposition will play an essential role in the proof of one of the main theorems of this work (Theorem 4.3.5).

## 2.2 Archimedean fields

The real closed fields have a behaviour which is very similar to that of the field of real numbers $I\!R$. More precisely, any first order formula defined with coefficients in $I\!R$, which is true over $I\!R$ remains true over any real closed field containing $I\!R$. (see 2.4.11). Nevertheless, there are important properties which cannot be formulated with the help of a first order formula. Another class of fields, which shares many properties with $I\!R$, is the class of archimedean fields.

**Definition 2.2.1** *An ordered field $(F, P)$ is called* **archimedean** *if every element of $F$ is bounded by a natural number, that is*

$$x \in F \Rightarrow \exists n \in I\!N : |x| \leq_P n \tag{1}$$

**Remark 2.2.2** We stress the fact that the notion of *archimedean field* depends on the given ordering. In general, a given field can have at the same time orderings for which it is archimedean and orderings for which it is not archimedean.

The next proposition gives a classification of archimedean fields.

**Proposition 2.2.3** *Let $(F, P)$ be an archimedean field. Then there is a unique injective homomorphism of rings $\phi : F \mapsto I\!R$ which respects the orderings.*

In other words, the archimedean fields are the subfields of $I\!R$ with the induced orderings.

**Corollary 2.2.4** *Archimedean fields are dense in $I\!R$.*

This follows from the fact that every archimedean field has characteristic 0, so it contains the field of rational numbers which is dense in $I\!R$.

We will need the following proposition:

**Proposition 2.2.5** *Let $(F, P)$ be an archimedean field. Let $\epsilon > 0$ be a rational number and $x \in F$ with $0 < x < 1$. Then there exists a natural number $M > 0$ such that $x^M < \epsilon$.*

## 2.3 Real varieties

We come to the important definition of a real variety. In contrary to the situation in Algebraic Geometric, in Real Algebraic Geometry questions of reducibility play a less important role. Similarly, it is sufficient to handle affine varieties, since real projective varieties are less needed in Real Algebraic Geometry. So we will define affine real varieties. We follow the presentation in [1].

Let $k$ be a field with algebraic closure $\overline{k}$. Let $K$ be a field with $k \subseteq K \subseteq \overline{k}$. For any subset $T$ of $k[t_1, \ldots, t_n]$ and any subset $V$ of $K^n$ we set

$$Z_K(T) := \{x \in K^n : \forall f \in T \ f(x) = 0\}$$

$$I_k(V) := \{f \in k[t_1, \ldots, t_n] : \forall x \in V \ f(x) = 0\} \tag{2}$$

**Definition 2.3.1** *An* **affine** $k$**-variety** *is a subset* $V$ *of* $\overline{k}^n$ *which has the form* $V = Z_{\overline{k}}(I)$ *for an ideal* $I \subseteq k[t_1, \dots, t_n]$*. For an intermediate field* $k \subseteq K \subseteq \overline{k}$ *we call* $V(K) := V \cap K^n$ *the* $K$**-rational points**. *If* $k = K = R$ *is a real closed field, we call* $V(R)$ *a* **real affine variety**.

**Remark 2.3.2**

a) It is possible to define real affine varieties in a more intrinsic way. This is done in [2] using sheaves and regular functions. This more general definition is compatible with our definition but will not be needed here.

b) The real affine varieties are exactly the algebraic subsets of the spaces $R^n$. In the following sections, we will denote a real affine variety by $V$. Since we do not consider other varieties, this should not yield to confusions.

**Definition 2.3.3** *Let* $V \subseteq \overline{k}^n$ *be an affine* $k$*-variety. The* **affine algebra** $k[V]$ *is the finitely generated, reduced algebra* $k[V] = k[t_1, \dots, t_n]/I_k(V)$*. It is also called the* **coordinate ring** *of* $V$*.*

If $R$ is a real closed field, then $R$ has a unique ordering which induces a topology on $R$ and hence on $R^n$. If $V \subseteq R^n$ is a real affine variety, then the topology of $R^n$ induces a topology on $V(R)$. This is a canonical topology and in the sequel all topological notions relate to it.

The second topology we will consider is the Zariski-topology on a real affine variety. It is defined in the usual way.

## 2.4 Semialgebraic sets

We will give the definition of a semialgebraic set. These sets are the central object in semialgebraic geometry and have a very rich structure and nice properties. One can see semialgebraic sets as the semialgebraic counterpart to algebraic sets in Algebraic Geometry.

We fix a real closed field $R$ and a positive integer $n$.

**Notation 2.4.1** Let $\{f > 0\}$ denote the set $\{x \in R^n : f(x) > 0\}$, where $f \in R[t_1, \dots, t_n]$. The sets $\{f = 0\}, \{f \geq 0\}, \{f_1 > 0, \dots, f_n > 0\}, \dots$ are defined in an analogous way.

**Definition 2.4.2** *A* **semialgebraic set** $S \subseteq R^n$ *is a finite boolean combination of sets of the form* $\{f > 0\}$ *where* $f \in R[t_1, \dots, t_n]$ *is a polynomial with* $n$ *variables.*

With other words, semialgebraic sets arise from sets of the form $\{x \in R^n : f(x) > 0\}$ by taking finitely many intersections, unions and complements. It is easy to show that every semialgebraic $S$ set can be written as a finite union of sets of the form

$$\{f = 0, g_1 > 0, \dots, g_n > 0\}$$

.

More generally, we can define semialgebraic sets in a real affine variety. Let $V$ be a real affine variety. Given $f \in R[V]$ and $x \in V$, the value $f(x) \in R$ is well defined. Now semialgebraic sets of $V$ are boolean combinations of sets of the form $\{f > 0\}$ where $f \in R[V]$. Note that this definition coincides with the definition above in the case $V = R^n$ and that the semialgebraic sets of $V$ are the restrictions of those of $R^n$ to $V$.

The most important properties of semialgebraic sets are given in the following proposition:

**Proposition 2.4.3** *Let $S \in R^n$ be a semialgebraic set. Then*

a) *The interior, the closure and the border of $S$ are semialgebraic.*

b) *The image of $S$ under the projection $\pi : R^n \mapsto R^{n-1}$, $\pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$ is semialgebraic. (Quantifier elimination or Tarski–Seidenberg–Principle)*

See [2] for proofs.

**Remark 2.4.4**

a) The semialgebraic subsets of $R$ are the finite unions of intervals and points.

b) Semialgebraic sets can be very difficult sets, however there are many properties of finiteness or triviality. See [8] for details.

**Definition 2.4.5** *Let $V$ be a real affine variety over $R$. A semialgebraic set $S \subseteq V$ is called* **basic open** *if it can be written as*

$$S = \{x \in V : f_1(x) > 0, \dots, f_m(x) > 0\} \tag{3}$$

*for a natural number $m$ and functions $f_1, \dots, f_m \in R[V]$. The notion of* **basic closed set** *is defined in an analogous way by relaxing the inequalities.*

Now we have to give an important notion from model theory. The reader is referred to [7] for details.

**Definition 2.4.6** *Let $A$ be a commutative ring with unit. A* **formula** $\Phi(x)$ **with parameters in** $A$ *(in the language of ordered fields with parameters in $A$) is an expression which is built up by a finite number of conjunctions, disjunctions, negations, universal and existential quantifiers on the variables starting from atomic formulae, which are the expressions $f(x) > 0$ where $f \in A[x], x = (x_1, \dots, x_n)$. A* **sentence** *is a formula without free variables.*

The part b) of the Proposition 2.4.3 is an equivalent version of the following proposition:

**Proposition 2.4.7** *A subset $S \subseteq R^n$ is semialgebraic if and only if there exists a formula $\Phi(x)$ with parameters in $R$ and free variables $x = (x_1, \ldots, x_n)$ such that $S = \{x \in R^n : \Phi(x)\}$.*

This proposition can be very useful to show that a given set is semialgebraic.

**Definition 2.4.8** *Let $S \subseteq R^n$ and $T \subseteq R^m$ be semialgebraic subsets. A function $f : S \mapsto T$ is called a* **semialgebraic function** *if the graph*

$$\Gamma = \{(x, y) \in S \times R^m : y = f(x)\}$$

*is a semialgebraic subset of $R^n \times R^m$.*

**Example 2.4.9** If $f : R^n \mapsto R^m$ is a polynomial function, then $f$ is a semialgebraic function.

**Proposition 2.4.10** *Let $S \subseteq R^n$ and $T \subseteq R^m$ be semialgebraic subsets and let $f : S \mapsto T$ be a semialgebraic function. Then the image $f(S)$ is a semialgebraic subset of $T$.*

There is another useful interpretation of the Tarski–Seidenberg–Principle:

**Proposition 2.4.11 (Model Completeness)** *Let $\Phi$ be a formula over $R$ without free variables and let $K$ be a real closed extension of $R$. Then $\Phi$ is true in $R$ if and only if it is true in $K$.*

See [2] for the proofs of these two propositions.

There is an important inequality found by Hörmander and Łojasiewicz (see [10], [2] or [1]) in order to attack questions arising from the theory of distributions. This inequality is very often used in Real Geometry to glue together different functions.

**Proposition 2.4.12 (Łojasiewicz' Inequality)** *Let $S$ be a semialgebraic, bounded and closed subset of a real affine $R$-variety; let $f$ and $g$ be two semialgebraic continuous functions $f, g : S \mapsto R$ such that the zeros of $f$ are zeros of $g$ too. Then there exist an integer $N > 0$ and a constant $c \in R$ such that*

$$|g|^N \leq c * |f| \tag{4}$$

*holds on $S$. In addition,we can achieve that this inequality is strict on the set $\{f \neq 0\}$.*

## 2.5 The real spectrum of a ring

Now, let us consider a commutativ ring A with unit 1.

**Definition 2.5.1** *An ideal $I$ is called* **real** *if for all $a_1, \ldots, a_n \in A$ with $a_1^2 + \ldots + a_n^2 \in I$ we have $a_1, \ldots, a_n \in I$.*

**Example 2.5.2** If $\emptyset \neq V \subseteq R^n$ is a real affine variety, then the ideal $I_R(V)$ is real.

**Definition 2.5.3** *The* **real spectrum of** *$A$ is the set of paires $\alpha = (\wp, T)$ where $\wp$ is a real prime ideal of $A$ and $T$ is an ordering of the residue field* $\mathrm{Quot}A/\wp$.

**Notation 2.5.4** We will denote by $\mathrm{Spec_r}A$ the real spectrum of $A$. $k(\alpha)$ denotes the real closure of $\mathrm{Quot}A/\wp$ with regard to the ordering $T$.

There are other interpretations of the real spectrum of a ring. For our purposes the next definition is important:

**Definition 2.5.5** *Let $A$ be a commutative ring with unit. A* **prime cone** *of $A$ is a subset $P \subseteq A$ such that*

*a)* $P + P \subseteq P$

*b)* $P \cdot P \subseteq P$

*c)* $P \cup (-P) = A$

*d)* $\mathrm{supp}\ P := P \cap (-P)$ *is a prime ideal of $A$.*

We can consider the real spectrum of a ring as the set of prime cones. The identification is as follows: if $\alpha = (\wp, T) \in \mathrm{Spec_r}A$, then $P := \{a \in A : \overline{a} \in T\}$ (where $\overline{a}$ denotes the image of $a$ in $\mathrm{Quot}A/\wp$) is a prime cone. If $P$ is a prime cone, then $P$ induces an ordering $\overline{P}$ on $\mathrm{Quot}A/\mathrm{supp}\ P$ and hence a point $(\mathrm{supp}\ P, \overline{P}) \in \mathrm{Spec_r}A$. We will use in the sequel the two interpretations, with the identification above all the results stated for prime cones can be reformulated in terms of prime ideals with orderings on the residue fields and vice versa. We remark that there is actually a third interpretation of the real spectrum as the set of non-trivial homomorphisms from $A$ into some real closed field modulo an equivalence relation. This will be less important for us. See [2] for details.

As a set, the real spectrum is not very interesting, so we have to provide it with the structure of a topological space. To begin with, we can see the elements of $A$ as functions on $\mathrm{Spec_r}A$. We have a canonical morphism (by abuse of notation also denoted by $\alpha$)

$$\alpha : A \mapsto \mathrm{Quot}A/\wp \hookrightarrow k(\alpha)$$

If $\alpha \in \mathrm{Spec_r}A$, $f \in A$, then $f(\alpha) := \alpha(f)$. So $f$ is a function on $\mathrm{Spec_r}A$, but the value fields varie with $\alpha$. Formally, we can say that $f$ is an application

$$f : \mathrm{Spec_r}A \mapsto \prod_{\alpha \in \mathrm{Spec_r}A} k(\alpha)$$

such that $f(\alpha) \in k(\alpha)$. The reader who is familiar with the (Zariski–) Spectrum of a ring should note the similarity with the real spectrum.

**Notation 2.5.6** Let $f \in A$. By $\{f > 0\}$ (resp. $\{f \geq 0\}$, ...) we denote the set $\{\alpha \in \mathrm{Spec_r} A : f(\alpha) > 0\}$ (resp. $\{\alpha \in \mathrm{Spec_r} A : f(\alpha) \geq 0\}$, ...).

**Definition 2.5.7** *The* **spectral topology** *is the topology on* $\mathrm{Spec_r} A$ *which has* $\{\{f > 0\} : f \in A\}$ *as a subbasis of open sets. It follows that a basis of open sets is given by* $\{\{f_1 > 0, \ldots, f_n > 0\} : n \in \mathbb{N}; f_1, \ldots, f_n \in A\}$.

**Remark 2.5.8** The spectral topology on $\mathrm{Spec_r} A$ is also called **Harrison-Topology**.

Later on, we will establish a relation between real affine varieties with semialgebraic sets and real spectra with a certain kind of sets, called constructible. They can be seen as the generalization of semialgebraic sets and there is a deep connection between these two notions.

**Definition 2.5.9** *A* **constructible set** *is a finite boolean combination of sets of the form* $\{f > 0\}$ *whith* $f \in A$. *The* **constructible topology** *on* $\mathrm{Spec_r} A$ *is the topology which has the set of constructible sets as an open basis.*

**Remark 2.5.10**

a) If not mentioned otherwise, any topological notion like open, closure etc. will always refer to the spectral topology on $\mathrm{Spec_r} A$. The constructible topology has many advantages in some proofs, but the more natural and interesting topology is the spectral one.

b) It is clear that the constructible topology is finer than the spectral one.

**Proposition 2.5.11** *Let $A$ be any commutative ring with unit.*

*a)* $\mathrm{Spec_r} A$ *is compact with respect to the constructible topology.*

*b) The sets which are open and closed for the constructible topology are exactly the constructible sets.*

*c) The spectral topology is quasicompact, but in general not Hausdorff.*

*d)* $\mathrm{Spec_r} A$ *is a $T_0$-space, that is, if $x, y \in \mathrm{Spec_r} A$, $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$ then $x = y$.*

**Notation 2.5.12** For $x, y \in \mathrm{Spec_r} A$ with $y \in \overline{\{x\}}$ we write $x \to y$. In this case, $y$ is called a **specialization** of $x$ and $x$ a **generalization** of $y$.

**Proposition 2.5.13** *Let $\phi : A \mapsto B$ be a homomorphism of commutative rings with unit, then we have an induced map $\phi^* : \mathrm{Spec_r} B \mapsto \mathrm{Spec_r} A$ defined by*

$$P \in \mathrm{Spec_r} B \mapsto \phi^{-1}(P) \in \mathrm{Spec_r} A$$

*This map is continuous with respect to both the Harrison- and the constructible topology.*

In other words, $\mathrm{Spec_r}$ is a contravariant functor from the category of commutative rings with unit to the category of topologic spaces.

Next, we consider an affine real variety $\emptyset \neq V \subseteq R^n$. We set

$$A := R[V] = R[t_1, \dots, t_n]/I_R(V) \tag{5}$$

We want to establish a relation between $V$ (equipped with the induced topology) and the topological space $\mathrm{Spec_r} A$ with the spectral topology.

Let $x \in V$. Then we have an application $\mathrm{ev}_x : A \twoheadrightarrow R$ defined by $\mathrm{ev}_x(f) := f(x)$. If $\wp$ denotes the kernel of this map, we have $\mathrm{Quot}A/\wp = R$ and $R$ is equipped with a unique ordering $T$. So $(\wp, T) \in \mathrm{Spec_r} A$. This defines an application $\phi : V \mapsto \mathrm{Spec_r} A$.

**Proposition 2.5.14** *$\phi$ is an injective and continuous map.*

We can view $V$ as a subspace of $\mathrm{Spec_r} A$, the topology of $V$ being induced by the spectral topology of $\mathrm{Spec_r} A$. The main interest in considering $\mathrm{Spec_r} A$ instead of $V$ is that on the one hand, $\mathrm{Spec_r} A$ reflects the properties of $V$, on the other, it has a certain number of advantages, e.g. it is quasicompact. As a first, very important result in this direction, we have:

**Proposition 2.5.15** *The boolean algebra of semialgebraic sets of $V$ and the boolean algebra of constructible sets of $\mathrm{Spec_r} A$ are isomorphic.*

We will describe this isomorphism more explicitely. If $C \subseteq \mathrm{Spec_r} A$ is a constructible set, then the corresponding semialgebraic set of $V$ is the set $V \cap C$. For a semialgebraic set $S \subseteq V$, the corresponding constructible set is $\tilde{S}$, the closure of $S$ in the constructible topology.

If a semialgebraic set $S \subseteq V$ is given by a boolean combination of sets $\{f > 0\} \subseteq V$ where $f \in A$, then the corresponding set in the real spectrum is the same combination of $\{f > 0\} \subseteq \mathrm{Spec_r} A$ and vice versa.

## 2.6 Dimension

**Definition 2.6.1** *Let $S \subseteq R^n$ be a semialgebraic set and $V$ its Zariski-closure. The* **dimension of** *$S$ is by definition the (Krull-) dimension of the ring $R[V]$, that is the maximal length of a chain of prime ideals in $R[V]$. We denote it by* $\dim S$

This definition is a generalization of the notion of *dimension* on a variety, which we can consider as a semialgebraic set defined by some equations.

**Proposition 2.6.2** *Let $S \subseteq R^n$ be a semialgebraic set. Then*

$$\dim S = \dim(\mathrm{adh}\ S) = \dim(\mathrm{adh}_Z(S)) \tag{6}$$

*where* adh *denotes the closure for the topology induced by the one of $R^n$ and* $\mathrm{adh}_Z$ *the closure for the Zariski–topology.*

**Proposition 2.6.3** *Let $S$ be a semialgebraic set and let $f : S \mapsto R^p$ be a semialgebraic function. Then* $\dim S \geq \dim f(S)$.

**Definition 2.6.4** *Let $A$ be a commutative ring and $C \subseteq \mathrm{Spec_r} A$ a constructible set. The dimension of $C$ is the maximal length $n$ of a chain $\alpha_n \subset \alpha_{n-1} \subset \ldots \subset \alpha_0$ which is contained in $C$. We denote it by* $\dim_c C$.

**Proposition 2.6.5** *Let $R$ be a real closed field and let $S \subseteq R^n$ be a semialgebraic set. Then*

$$\dim S = \dim_c \tilde{S} \tag{7}$$

See [2] for a proof.

**Definition 2.6.6** *Let $A$ be a commutative ring. For $\alpha \in \mathrm{Spec_r} A$, the dimension of $\alpha$, denoted by $\dim_r \alpha$ is by definition the (Krull-) dimension of the ring $A/\mathrm{supp}\ (\alpha)$. The dimension of a constructible set $\emptyset \neq C \subseteq \mathrm{Spec_r} A$ is the maximum of the dimensions $\dim \alpha$ with $\alpha \in C$. We denote it by $\dim_r C$. The real dimension of the ring $A$ ($\dim_r A$) is the dimension of $\mathrm{Spec_r} A$.*

**Proposition 2.6.7** *Let $A$ be a commutative ring and let $C \subseteq \mathrm{Spec_r} A$ be a constructible set. Then*

$$\dim_c C \leq \dim_r C \tag{8}$$

**Definition 2.6.8** *Let $k$ be a field and let $A$ be a $k$-algebra. Then the (transcendental-) dimension of $A$, denoted by $\dim_t A$, is the maximal number $n$ of elements $a_1, \ldots, a_n$ of $A$ such that there is no polynomial $0 \neq p \in k[t_1, \ldots, t_n]$ with $p(a_1, \ldots, a_n) = 0$.*

**Proposition 2.6.9** *Let $R$ be a real closed field and let $A$ be a $R$-algebra. Then*

$$\dim_r A \leq \dim_t A \tag{9}$$

## 2.7 The main results

The central subject of this work is the following theorem:

**Theorem 2.7.1 (Bröcker-Scheiderer)** *Let $R$ be a real closed field and let $V$ be a real affine $R$-variety of dimension $d > 0$. Given $n \geq d$ functions $f_1, \ldots, f_n \in R[V]$ then there are functions $g_1, \ldots, g_m \in R[V]$ with $m \leq d$ such that*

$$\{x \in V : f_1(x) > 0, \ldots, f_n(x) > 0\} = \{x \in V : g_1(x) > 0, \ldots, g_m(x) > 0\} \tag{10}$$

**Remark 2.7.2** One defines an invariant $s(V)$, which depends only on $V$: $s(V)$ is the smallest natural number (or $\infty$) such that every basic open set can be written with only $s(V)$ functions. Theorem 2.7.1 states that $s(V) \leq \dim V$ for a $R$-variety over a real closed field $R$ (one even has equality). The number $s(V)$ is called the **stability index** of $V$.

In the proofs of Theorem 2.7.1, one uses much theoretical machinery, like

- quadratic forms and the theorem of Tsen-Lang (see [15])
- theory of fans, spaces of signs and spaces of orderings (see [3])
- theory of real valuations and real places (see [4],[5],[6])

These approaches have the inconvenient that they are not very constructive (apart from Mahé's proof where the solution of algebraic equations over function fields is necessary which could be difficile in practice).

The aim of this work is to present theorems in two directions:

- Obtaining some constructions for special cases (see chapter 3 and 4).
- Obtaining information about the dependance of the functions $g_i$ from the functions $f_i$ (see chapter 5)

For the first purpose we don't admit Theorem 2.7.1, whereas we use it for the second purpose.

Let us state Theorem 2.7.1 in the real spectrum of a ring.

**Theorem 2.7.3** *Let $A$ be any $R$-algebra of transcendence degree $d > 0$ over the real closed field $R$, then any basic open set in the real spectrum of $A$ can be written with only $d$ inequalities.*

See [15] for a proof.

**Theorem 2.7.4 (The "t-invariant")** *Let $R$ be a real closed field and let $V$ be a real affine variety of dimension $d > 0$ over $R$. Then there exists an invariant $t(V)$ which depends only on $V$, such that every open semialgebraic set of $V$ can be written as the union of at most $t(V)$ basic open sets. This invariant is bounded by*

$$t(V) \leq (d+1) * \tau(s(V)) \tag{11}$$

*where the function $\tau$ is defined by*

$$\tau(s) = \begin{cases} s & \text{for } s \leq 2 \\ \begin{pmatrix} 4^{s-1} - 2^{s-1} + 1 \\ 2 * 4^{s-2} - 2^{s-2} + 1 \end{pmatrix} & \text{for } s \geq 3 \end{cases} \tag{12}$$

**Theorem 2.7.5** *Let $R$ be a real closed field and let $V$ be a real affine variety of dimension $d > 0$ over $R$. Then every semialgebraic set of $V$ can be written in the form*

$$S = \bigcup_{i=1}^{t} \{g_i = 0, g_{i,1} > 0, \ldots, g_{i,s} > 0\} \tag{13}$$

*with $g_i, g_{i,j} \in R[V]$, $t \leq (d+1) * \tau(d)$ and $s \leq d$.*

**Theorem 2.7.6** *Let $R$ be a real closed field and let $V$ be a real affine variety of dimension $d > 0$ over $R$. Then every semialgebraic basic closed set of $V$ can be written in the form*

$$S = \{g_1 \geq 0, \ldots, g_{\overline{s}} \geq 0\} \tag{14}$$

*with $g_i \in R[V]$ and $\overline{s} \leq \frac{d(d+1)}{2}$.*

**Theorem 2.7.7** *Let $R$ be a real closed field and let $V$ be a real affine variety of dimension $d > 0$ over $R$. Then every closed semialgebraic set of $V$ can be written in the form*

$$S = \bigcup_{i=1}^{\overline{t}} \{g_{i,1} \geq 0, \ldots, g_{i,\overline{s}} \geq 0\} \tag{15}$$

*with $g_i \in R[V]$, $\overline{t} \leq d^{(d+1)\tau(d)}$ and $\overline{s} \leq \frac{d(d+1)}{2}$.*

For a proof of these theorems, see [3].

# 3 Convex Interpolation

In [9] it is shown in which way some methods of interpolation and approximation can be used to reduce the number of inequalities in the description of bounded convex polygons of $I\!R^2$. The result is a constructive proof of the fact that every such polygon can be described by 3 inequalities. With the help of a generalization of the used arguments we will give a similar proof which leads to a description with only 2 instead of 3 inequalities.

## 3.1 Theorem of Convex Interpolation

In this chapter, we will show a useful theorem which will give us some information about the two-dimensional case of the Bröcker-Scheiderer-Theorem. Since the Convex Interpolation could be useful for other applications than Real Geometry, we give the proof in the general case, although we only use a special case, namely $n = 2$.

**Definition 3.1.1** *Given $m$ points $y_1, \ldots, y_m \in I\!R^n$, we will say that they lie in a* **convex position***, if no point lies in the convex hull of the others.*

**Theorem 3.1.2** *Given $m$ points $y_1, \ldots, y_m \in I\!R^n$ in convex position, there is a polynomial $p \in I\!R[x_1, ..., x_n]$ with the following properties:*

- *The sets $\{x \in I\!R^n : p(x) \geq 0\}$ and $\{x \in I\!R^n : p(x) > 0\}$ are convex.*
- *The points $y_1, \ldots, y_m$ lie on the border of this set.*

**Remark 3.1.3** In [12], a similar theorem is given in the one-dimensional case. Our case is in some sense the multi-dimensional generalization of this theorem.

For the proof, we need the following lemma:

**Lemma 3.1.4** *For a given $\epsilon > 0$, $y_1, \ldots, y_m \in I\!R^n$ in convex position and $i \in \{1, \ldots, m\}$ there is a polynomial $p_i \in I\!R[x_1, \ldots, x_n]$ with:*

- $p_i(y_i) = 1$
- $\mid p_i(y_j) \mid < \epsilon$ *for* $j \neq i$
- *The function $p_i$ on $I\!R^n$ is convex.*

**Proof:** Using the convex position of the points $y_1, \ldots, y_m$, we find a linear function $g : I\!R^n \mapsto I\!R$ such that $g(y_i) > 0$ and $g(y_j) < 0$ for $j \neq i$. With the help of a linear transformation we can assume that $g(y_i) = 1$ and $-1 < g(y_j) < 1$ for $j \neq i$. For a suitable exponent $2s$, all the values $g(y_j)^{2s}$ for $j \neq i$ are of absolute value $< \epsilon$ and the function $p_i(x) := g(x)^{2s}$ is convex, so the Lemma is proven. $\square$

Let $\epsilon > 0$ be a real number that will be fixed later on. For each $y_i$ we choose a polynomial $p_i$ as in the Lemma. We write

$$p = 1 - \left( \sum_{i=1}^{m} c_i p_i \right) \tag{16}$$

where the $c_i$ are positive real numbers which are to be found. The conditions $p(y_j) = 0$ for $j = 1, \ldots, m$ yield to the system of equations:

$$\sum_{i=1}^{m} c_i p_i(y_j) = 1 \tag{17}$$

With the help of matrices this can be written as

$$\begin{pmatrix} p_1(y_1) & \ldots & p_m(y_1) \\ \vdots & \vdots & \vdots \\ p_1(y_m) & \ldots & p_m(y_m) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} \tag{18}$$

If we write C for $\begin{pmatrix} c_1 \\ \vdots \\ \vdots \\ c_m \end{pmatrix}$ ,1 for $\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$ and $I$ for the matrix unity, the condition is

$$(I + A)C = 1 \tag{19}$$

where A is a matrix of which the entries on the diagonal are zero and the others of absolute value smaller than $\epsilon$. We consider in $\mathbb{R}^m$ the maximum norm, that is

$$\left\| \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \right\| = \max_{i=1}^{m} | a_i | \tag{20}$$

We also consider the associated matrix norm. Is $A = (a_{ij})_{i=1,\ldots,m; j=1,\ldots,m}$ so

$$\| A \| = \max_{j=1}^{m} \sum_{i=1}^{m} | a_{ij} | \leq (m-1)\epsilon \tag{21}$$

We choose $0 < \epsilon < \frac{1}{2(m-1)}$, consequently we have $\| A \| < \frac{1}{2}$. Now, we consider the equation

$$(I + A)C = 1 \tag{22}$$

For $\| A \| < \frac{1}{2} < 1$, $(I + A)$ is invertible, hence there is a unique vector C that fulfils this equation. An elementary calculation shows that

$$\| C - 1 \| \leq \frac{\| A \|}{1 - \| A \|} < 1 \tag{23}$$

From the definition of the norm we conclude that $| c_i - 1 | < 1$, hence that $c_i > 0$ for $i = 1, \ldots, m$.

The function $p$ constructed in this way is the function we sought in Theorem 3.1.2: It vanishes on the $y_i$ by construction and it is a linear combination with negative coefficients of convex functions, hence it is a concave function. Consequently, the sets $\{x \in I\!R^n : p(x) \geq 0\}$ and $\{x \in I\!R^n : p(x) > 0\}$ are convex. This finishes the proof. $\Box$

## 3.2 Application to the Bröcker-Scheiderer-Theorem

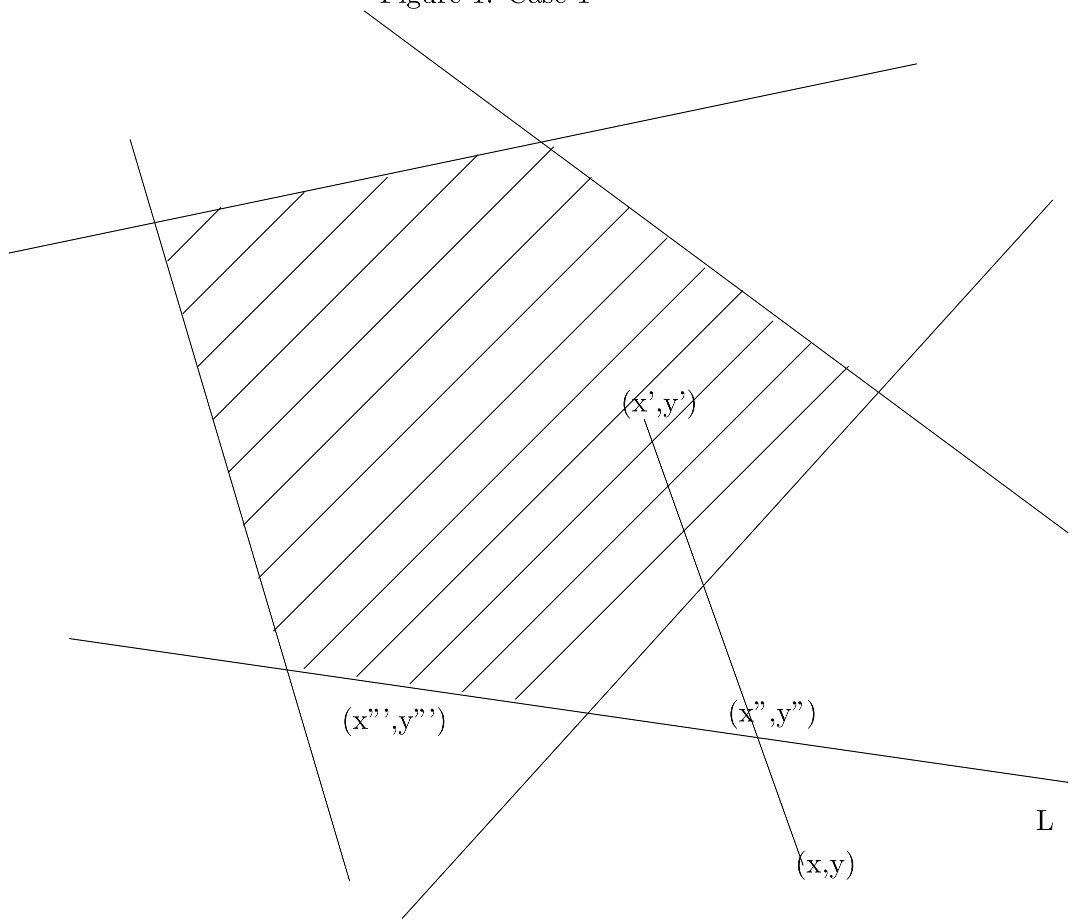
In this section we want to apply Theorem 3.1.2 to the reduction of inequalities in 2 variables. More precisely, we want to handle a special case of Theorem 2.7.1, namely $V = I\!R^2$ and all the functions $f_i$ are linear functions. Theorem 2.7.1 states that we can describe every basic open set in $I\!R^2$ with only two functions, but it gives no information on how to find them. With the help of the preceding section, we can give a construction.

For this, we consider $m$ linear polynomials $f_1, \ldots, f_m \in I\!R[x, y]$. The set described by these functions, that is the set $S := \{x \in I\!R^2 : f_1(x) > 0, \ldots, f_n(x) > 0\}$, is the intersection of open convex sets (namely the halfplanes $\{f_i(x) > 0\}$) and hence convex. If $S$ is the empty set or $I\!R^2$, then it can be described by a single inequality, we suppose henceforth that this is not the case. We have to consider two cases:
Case 1: $S$ is bounded.
Case 2: $S$ is unbounded.

**Case 1:** We suppose $S$ is bounded. Then $S$ is a convex polygon whose sides are given by some of the functions $f_i$. Without loss of generality, we can assume that in the description $S = \{x \in I\!R^2 : f_1(x) > 0, \ldots, f_n(x) > 0\}$ the sides of the polygon $S$ are exactly the lines defined by the equations $f_i = 0$. We set $f := \prod_i f_i$. To find the second function, we apply Theorem 3.1.2 to the polygon $S$. Hence, we find a polynomial function $g \in I\!R[x, y]$ such that $g$ vanishes on the vertices of $S$ and is positive in the interior of $S$ and such that the set $\{(x, y) \in I\!R^2 : g(x, y) > 0\}$ is convex. Now we will show that $S = \{(x, y) \in I\!R^2 : f(x, y) > 0, g(x, y) > 0\}$. Consider $(x, y) \in I\!R^2$. If $(x, y)$ is in the interior of $S$, then all the values $f_i(x, y)$ and hence $f(x, y)$ is positive and $g(x, y) > 0$ too. If $(x, y)$ lies on a line defined by the functions $f_i$, then $f(x, y) = 0$. If an odd number of the values $f_i(x, y)$ is negative, then $f(x, y) < 0$. So it remains the case that an even number of the values $f_i(x, y)$ is negative. See the figure.

Consider a point $(x', y')$ in the interior of $S$ such that the line segment passing through $(x, y)$ and $(x', y')$ does not meet any point of intersection of two or more lines defined by the functions $f_i$. This is possible since the number of such points

Figure 1: Case 1




of intersection is finite. Then the line segment intersects an even number of times a zero-set of one of the functions $f_i$. We consider the point of intersection which is the nearest to $(x, y)$, we call it $(x'', y'')$ and the corresponding line $L$. Since at least two of the values $f_i(x, y)$ are negative, at least one of the values $f_i(x'', y'')$ is negative and $(x'', y'')$ is not on the border of $S$. Assume that $g(x, y) > 0$. Then, by convexity of the set $\{g > 0\}$ and the fact that $g(x', y') > 0$ and $(x'', y'')$ is between these two points, $g(x'', y'') > 0$. The restriction of $g$ to $L$ gives a polynomial function on $L$, $g|_L$, which has two zeros, namely the two vertices $L$ passes through. On the other hand, $g|_L(x'', y'') = g(x'', y'') > 0$, hence $g|_L$ is not identically 0. So we can choose a point $(x''', y''')$ between the two vertices on $L$ such that $g(x''', y''') \neq 0$. Since $g > 0$ in the interior of $S$, $g(x''', y''') > 0$. But one of the two vertices on $L$ lies between $(x'', y'')$ and $(x''', y''')$ and the set $g|_L > 0$ is a convex set on this line, this gives a contradiction to the fact that $g = 0$ on this vertice.

**Case 2:** $S$ is unbounded. In this case, we can choose a line $L$ which does not intersect the closure of $S$. We consider for the moment $I\!R^2$ as a subspace of $P^2$, the projective space of dimension 2. $P^2$ can be considered as $I\!R^3/\sim$ where $(x,y,z) \sim (x',y',z') \Leftrightarrow \exists\lambda \neq 0 : (x,y,z) = \lambda(x',y',z')$. The identification is then

$$(x,y) \in I\!R^2 \mapsto (x,y,1) \in P^2$$

The line at infinity is the line $z = 0$. We denote by $A$ the intersection of the line $L$ with the line at infinity. We choose a point $B \neq A$ on $L$ and a point $C \neq A$ at the line at infinity. These three points do not lie on a line, so we find a projective change of coordinates $T : P^2 \mapsto P^2$ which satisfy the following conditions:

a) $T(A) = A$

b) $T(B) = C$

c) $T(C) = B$

$T$ is induced by a bijective map $T : I\!R^3 \mapsto I\!R^3$ of the form

$$T(x,y,z) = (a_1x + b_1y + c_1z, a_2x + b_2y + c_2z, a_3x + b_3y + c_3z) \tag{24}$$

where the $a_i, b_i, c_i$ are in $I\!R$. The proof of this basic fact can be found in any book on Projective Geometry.

We return to the original situation, that is $I\!R^2$. If we remove the line $L$, then $T$ induces a map $T : I\!R^2 - L \mapsto I\!R^2 - L$. Since $L$ does not intersect the closure of $S$, the line at infinity does not intersect the closure of $T(S)$ which means that $T(S)$ is bounded. The set $S$ contains no point at infinity, so $T(S)$ and $L$ are disjoint sets. But $T$ maps lines into lines, so $T(S)$ is a bounded, open, convex polygon of $I\!R^2$.

By case 1, we can describe $T(S)$ with two polynomial functions $f, g \in I\!R[x,y]$. We have for $(x,y) \in S$

$$\begin{aligned} &T(x,y,1) = (a_1x + b_1y + c_1, a_2x + b_2y + c_2, a_3x + b_3y + c_3) \\ &\sim (\frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3}, 1) \end{aligned} \tag{25}$$

We conclude that

$$\begin{aligned} &S = \{(x,y) \in I\!R^2 - L : f\left(\frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3}\right) > 0, \\ &g\left(\frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3}\right) > 0\} \end{aligned} \tag{26}$$

Note that the line $a_3x + b_3y + c_3$ is exactly our line $L$. Multiplying $f$ and $g$ by sufficiently large even powers of $a_3x + b_3y + c_3$ we find polynomials $f', g' \in I\!R[x,y]$ such that

$$S = \{(x,y) \in I\!R^2 : f'(x,y) > 0, g'(x,y) > 0\} \tag{27}$$

This finishes the case 2. $\Box$

Hence we have proven directly the following theorem:

**Theorem 3.2.1** *Given linear polynomials* $f_1, \ldots, f_n \in I\!R[x, y]$, *then there are two polynomials* $f, g \in I\!R[x, y]$ *such that*

$$\{(x, y) \in I\!R^2 : f_1(x, y) > 0, \ldots, f_n(x, y) > 0\}$$

$$= \{(x, y) \in I\!R^2 : f(x, y) > 0, g(x, y) > 0\}$$

**Remark 3.2.2** This theorem is only a very special case of the Theorem of Bröcker-Scheiderer (Theorem 2.7.1). However, until now no direct proof of this simple case was known. In addition, our proof gives an algorithm for the reduction of linear inequalities. One may ask if with the help of the same techniques one can deal with convex polygons in higher dimensions. It turns out that other means are necessary, see Chapter 4.

# 4 Special cases of the Theorem of Bröcker-Scheiderer

## 4.1 Reductions with the help of power sums

Let $V \subseteq I\!R^p$ be a real affine variety, equipped with the induced topology. We set $x = (x_1, \ldots, x_p)$.

**Theorem 4.1.1** *Given $n$ functions $f_i \in I\!R[x], i = 1, \ldots, n$. We set*

$$P = \{x \in V : f_i(x) < 0 \textit{ for all } i\} \tag{28}$$

*We suppose, that the following three conditions hold:*

- *$P$ is bounded.*
- *In a point on the border of $P$ there are exactly one or exactly two among the functions $f_i$ which are $0$, in the latter case we call this point a vertex.*
- *There are only finitely many vertices.*

*Then there is an equivalent system of only two functions $f, g \in I\!R[x]$ i.e.*

$$\{x \in V : f_i(x) < 0 \textit{ for all } i\}$$

$$= \{x \in V : f(x) < 0, g(x) < 0\} \tag{29}$$

**Remark 4.1.2** The idea behind the proof is the following: let $x_1, \ldots, x_n$ be positive real numbers, then the power sum $x_1^M + \ldots + x_n^M$ with $M$ very large will be large if one of the $x_i$'s is greater than 1 and small if all the $x_i$'s are smaller than 1.

**Proof of Theorem 4.1.1:**

**Step 1:**
We will show that we can replace the system by an equivalent one for which the following property (*) holds:

*In the interior of $P$, all functions are between $-1$ (including) and $0$ and if a function is not $0$ in a vertex, then it is $-1$.* (*)

Since $P$ is bounded, there is for each function $f_i$ a constant $M < 0$, such that $f_i$ is bounded from below by $M$ on $P$. Replacing $f_i$ by $\frac{f_i}{-M}$ yields to an equivalent system for which the first part of the statement (*) holds.

Now we use the following lemma:

**Lemma 4.1.3** *Given $s$ real numbers $0 < \epsilon_i < 1, i = 1, \ldots, s$ there is a polynomial $p \in I\!R[x]$ with the following properties:*

- $x < 0 \Longrightarrow p(x) < 0, x > 0 \Longrightarrow p(x) > 0, p(0) = 0$

- $-1 \leq x < 0 \Longrightarrow -1 \leq p(x) < 0$
- *For* $i = 1, \ldots, s$ *we have* $p(-\epsilon_i) = -1$.
- $p(-1) = -1$

**Proof:** We set

$$p(x) := \frac{\prod_i (x+\epsilon_i)^2 * (x+1)^{2m+1}}{\prod_i \epsilon_i^2} - 1 \tag{30}$$

where $m$ is a natural number which we will fix later on.

- For $x \geq 0$, $p$ is strictly increasing, hence for all $x > 0$ we have $f(x) > f(0) = 0$.
- Obviously $p(-\epsilon_i) = -1$.
- For $-1 \leq x < 0$ we have $p(x) > -1$

The only condition to be verified is $-1 < x < 0 \Longrightarrow p(x) < 0$. We show that this is the case for $m$ sufficiently large.

First of all let $m = 0$. A simple calculation shows that $p'(0) = \sum_i \frac{2}{\epsilon_i} + 1 > 0$. Consequently there is an $\epsilon$ with $0 < \epsilon < 1$, such that for $-\epsilon < x < 0$ the inequality $p(x) < p(0) = 0$ holds. Let $M$ be the maximum of the continuous function $\frac{\prod_i (x+\epsilon_i)^2}{\prod_i \epsilon_i^2}$ on the intervall $[-1, -\epsilon]$. Obviously $M > 0$.

Now we choose $m$ sufficiently large such that

$$(1-\epsilon)^{2m+1} < \frac{1}{M} \tag{31}$$

This is possible since $0 < 1 - \epsilon < 1$. The polynomial $p$ corresponding to this $m$ satisfies the assertion $-1 < x < 0 \Longrightarrow p(x) < 0$: If $x \in [-1, -\epsilon]$, then

$$p(x) = \frac{\prod_i (x+\epsilon_i)^2 * (x+1)^{2m+1}}{\prod_i \epsilon_i^2} - 1 \leq M * (1-\epsilon)^{2m+1} - 1$$

$$< M * \frac{1}{M} - 1 = 0 \tag{32}$$

On the other hand, if $-\epsilon < x < 0$, then $p(x) < 0$ already for $m = 0$ and since $0 < 1 + x < 1$ this also remains true for a bigger $m$. □

Now, we consider one of the function, e.g. $f_i$. In the vertices of $P$, $f_i$ takes the value 0 or $-1$ or a value between $-1$ and 0. Let us denote the negative values $\neq -1$ of $f_i$ in the vertices by $-\epsilon_1, \ldots, -\epsilon_s$. By the Lemma 4.1.3 there is a polynomial $p \in I\!R[x]$, which takes for all the $-\epsilon_i$ the value $-1$. We replace $f_i$ by $p(f_i)$. This does not change the described set because $f_i$ and $p(f_i)$ have always the same sign. Since $f_i$ takes on $P$ values between $-1$ (including) and 0, $p(f_i)$ also takes on $P$ values between $-1$ (including) and 0. We execute this replacement for each function and consequently we get an equivalent system for which the statement (*) holds. □

**Step 2:**
We show a general proposition which could be also interesting for other applications.

**Proposition 4.1.4** *Let $S \subseteq \mathbb{R}^p$ be a bounded semi-algebraic subset of $\mathbb{R}^p$ with closure $X$, let $m, n$ be natural numbers with $1 \leq m \leq n$ and let $g_1, \ldots, g_n \in \mathbb{R}[x_1, \ldots, x_p]$ be polynomials in $p$ indeterminates. Suppose that the following conditions hold:*

- *If $x \in S$ then $0 \leq g_i(x) < 1$ for $i = 1, \ldots, m$.*
- *If $x \in \partial S = X - S$ then at most $m$ of the values $g_i(x)$ equal $1$ and if this is the case, the other values in this point are $0$.*
- *There are only finitely many points on the border of $S$ where exactly $m$ of the functions take the value $1$.*

*Then there is a natural number $M$ such that for all $x \in S$ we have:*

$$g_1(x)^M + \ldots + g_n(x)^M < m \tag{33}$$

**Proof:** We show first, that for each point $x$ on the border of $S$, for which exactly $m$ of the values $g_1(x), \ldots, g_n(x)$ are 1, there is a natural number $M$ and a neighbourhood in $S$ such that the inequation 33 is fulfilled in this neighbourhood.

**Lemma 4.1.5** *Let $A > 0$ be a real number, $m \geq 1$ a natural number and*

$$h(z_1, \ldots, z_m) := z_1 + \ldots + z_m + A * ((1 - z_1)^2 + \ldots + (1 - z_m)^2)^2$$

*Then there is a real number $\epsilon > 0$, such that:*

$$\forall i \quad 1 - \epsilon < z_i < 1 \Longrightarrow h(z_1, \ldots, z_m) < m \tag{34}$$

**Proof:** $h(1, \ldots, 1) = m$ and $\frac{\partial h}{\partial z_i}(1, \ldots, 1) = 1 > 0$ for all $i$, with the standard methods of analysis (mean value theorem) the assertion is straightforward. $\square$

**Lemma 4.1.6** *Let $S$ be as in Proposition 4.1.4 and let $x_0$ be a point on the border $\partial S$ such that exactly $m$ of the functions $g_1, \ldots, g_n$ are 1 on $x_0$. Then there is a number $M$ and an open subset $U \subseteq \mathbb{R}^p$ with $x_0 \in U$ such that $g_1(x)^M + \ldots + g_n(x)^M < m$ for all $x \in U \cap S$.*

**Proof:** We may suppose that $g_1(x_0) = \ldots = g_m(x_0) = 1$ and $g_{m+1}(x_0) = \ldots = g_n(x_0) = 0$. Let $g := (1 - g_1)^2 + \ldots + (1 - g_m)^2$. By assumption, for every $x \in X$ with $g(x) = 0$ we also have $g_i(x) = 0$ for any $i \in \{m+1, \ldots, n\}$. From Łojasiewicz's inequality (2.4.12) it follows that for every $i \in \{m + 1, \ldots, n\}$ there is a natural number $M_i$ and a real positive constant $A_i$ such that

$$g_i(x)^{M_i} \leq A_i g(x)^2 \tag{35}$$

for all $x \in X$. Let $M$ be the maximum of the $M_i$ and $A$ the sum of the $A_i$. Then for all $x \in S$

$$g_{m+1}(x)^M + \ldots + g_n(x)^M \leq g_{m+1}(x)^{M_{m+1}} + \ldots + g_n(x)^{M_n}$$

$$\leq A_{m+1} g(x)^2 + \ldots + A_n g(x)^2 = A g(x)^2 \tag{36}$$

It follows from Lemma 4.1.5 that there is a real number $\epsilon > 0$ such that if $1 - \epsilon < g_i(x) < 1$ for all $i = 1, \ldots, m$ then

$$g_1(x) + \ldots + g_m(x) + A * ((1 - g_1(x))^2 + \ldots + (1 - g_m(x))^2)^2 < m \tag{37}$$

We set

$$U := \{x \in R^p : 1 - \epsilon < g_i(x) < 1 + \epsilon \text{ for all } i = 1, \ldots, m\} \tag{38}$$

Then $M$ and $U$ satisfy the assertion of the Lemma:

- $U$ is open since the functions $g_1, \ldots, g_m$ are continuous.
- Since $g_1(x_0) = \ldots = g_m(x_0) = 1$, $x_0 \in U$.
- If $x \in U \cap S$, then $1 - \epsilon < g_i(x) < 1$ for $i = 1, \ldots, m$, hence

$$g_1(x)^M + \ldots + g_m(x)^M + g_{m+1}(x)^M + \ldots + g_n(x)^M$$

$$\leq g_1(x) + \ldots + g_m(x) + A * ((1 - g_1(x))^2 + \ldots + (1 - g_m(x))^2)^2$$

$$< m \tag{39}$$

**Continuation of the proof of Proposition 4.1.4:** Let $z_1, \ldots, z_s$ be the points on the border of $S$, for which exactly $m$ of the functions take the value 1. By the Lemma there are natural numbers $M_1, \ldots, M_s$ and open subsets of $R^p$, $U_1, \ldots, U_s$, such that for $x \in U_i \cap S$ we have $g_1(x)^{M_i} + \ldots + g_n(x)^{M_i} < m$ Now let $Y := X - \cup_i U_i$. Then $Y$ is a closed and bounded subset of $I\!R^p$, hence compact.

By construction, in each point of $Y$ there are at most $m-1$ of the functions $g_1, \ldots, g_n$ which equal 1 on this point and all the other functions are smaller than 1. Since $Y$ is compact, we find a real number $0 < \delta < 1$ such that in each point $x \in Y$ there are at least $n - m + 1$ among the values $g_1(x), \ldots, g_n(x)$ which are smaller than $\delta$.

We choose a natural number $M_Y$ such that

$$(n - m + 1) * \delta^{M_Y} < 1 \tag{40}$$

For $x \in Y$ we have

$$g_1(x)^{M_Y} + \ldots + g_n(x)^{M_Y} < m \tag{41}$$

since a sum of $n - m + 1$ summands is smaller than 1 and the other $m - 1$ summands are less or equal 1.

Now let $M$ be the maximum of the $M_i$ and $M_Y$. Then for $x \in S$

$$g_1(x)^M + \ldots + g_n(x)^M < m \tag{42}$$

since if $x \in U_i$ for a $i = 1, \ldots, s$, then

$$g_1(x)^M + \ldots + g_n(x)^M \leq g_1(x)^{M_i} + \ldots + g_n(x)^{M_i} < m \tag{43}$$

whereas if $x$ is in none of the $U_i$, then $x \in Y$, hence

$$g_1(x)^M + \ldots + g_n(x)^M \leq g_1(x)^{M_Y} + \ldots + g_n(x)^{M_Y} < m \tag{44}$$

This completes the proof of Proposition 4.1.4. □

**Remark 4.1.7** Obviously, we can increase the $M$ of the Proposition 4.1.4 and assume, for instance, that $M$ is even.

We apply this proposition with $S := P$, $m := 2$ and $g_i := f_i + 1$. Then the assumptions are satisfied: By (*) we have $0 \leq g_i < 1$ on $S$. Since in a point on the border of $S = P$ there are at most two of the functions $f_i$ that take the value 0, in each point of this border at most two of the functions $g_i$ are 1. The points on the border where exactly two of the functions $g_i$ take the value 1 correspond to the vertices of $S$ and their number is finite by assumption. Also by Step 1 the other $n-2$ functions among the $g_i$ take in such a vertex the value 0.

Proposition 4.1.4 gives us a natural number $M$ such that:

$$x \in P \Longrightarrow g_1(x)^M + \ldots + g_n(x)^M < 2 \tag{45}$$

**Step 3:**
Now we are able to prove Theorem 4.1.1. We set

$$f := g_1^M + \ldots + g_n^M - 2 \tag{46}$$

$$g := (-1)^{n+1} * \prod_i f_i \tag{47}$$

These two functions will describe $P$. If $x \in P$, then $f(x) < 0$ by Inequality 45. Since all the functions $f_i$ are negative in $x$, we have $g(x) < 0$. Conversely, take a point $x \in V$ with $f(x) < 0, g(x) < 0$. We have to show $x \in P$. But if this is not the case, then not all of the values $f_i(x)$ are negative. Since $g(x) < 0$, there are at least two of the values $f_i(x)$ which are strictly positive. Hence two of the values $g_i(x)$ are greater than 1 and consequently $f(x) > 0$, a contradiction which proves Theorem 4.1.1. □

### 4.2 A remark in the case of the plane $I\!R^2$

We consider the case $V = I\!R^2$ in the Theorem 4.1.1. We will show that the condition that there is only a finite number of vertices, is not a restriction. For this, we show that we can change the system into an equivalent one, where this statement is true.

**Definition 4.2.1** *Two systems $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_m\}$ of polynomials in two indeterminates are called* **equivalent***, if*

$$\{(x,y) \in I\!R^2 : g_1(x,y) > 0, \ldots, g_m(x,y) > 0\} =$$

$$\{(x,y) \in I\!R^2 : f_1(x,y) > 0, \ldots, f_n(x,y) > 0\} \tag{48}$$

**Proposition 4.2.2** *Given a system $\{f_1, \ldots, f_n\}$ of real polynomials ($\in I\!R[x,y]$). Then there is an equivalent system $\{g_1, \ldots, g_n\}$ of real polynomials ($\in I\!R[x,y]$), such that any two among the $g_i$ are without common divisor.*

**Proof:**
We consider the set of pairs $(i,j)$ with $i,j \in \{1,\ldots,n\}$ and $i < j$. On this set we introduce the lexicographic ordering, that is, $(i,j) \leq (i',j')$ iff $i < i'$ or ($i = i'$ and $j \leq j'$). Hence $(1,2) < (1,3) < \ldots < (1,n) < (2,3) < \ldots, (n-1,n)$. We call a system of $n$ polynomials $g_1, \ldots, g_n$ $(i,j)$*–good*, if for all pairs $(i',j') < (i,j)$ the polynomials $g_{i'}$ and $g_{j'}$ are without common divisor (wcd). The set of polynomials we search is a $(n-1,n)$-good system, for which in addition $g_{n-1}$ and $g_n$ are wcd. Among all the systems $g_1, \ldots, g_n$ which are equivalent to $f_1, \ldots, f_n$ we choose one which is $(i,j)$-good with $(i,j)$ maximal. We will show that then there is an equivalent system where in addition $g_i$ and $g_j$ are wcd. If $(i,j) = (n-1,n)$, we are ready. Otherwise we find a $(i,j)^+$-good system, where $(i,j)^+$ denotes the succesor for our ordering. This would be a contradiction. We show the following statement $(*)$:

To a given $(i,j)$-good system, which is equivalent to $f_1, \ldots, f_n$, there exists an equivalent $(i,j)$-good system such that in addition $g_i$ and $g_j$ are wcd. $\quad (*)$

Firstly, we can suppose that in all decompositions of the polynomials $g_1, \ldots, g_n$ in irreducible polynomials only first or second powers appear. This is clear from the fact that $x^3$ and $x$ have always the same sign. Let $d$ be the greatest common divisor of $g_i$ and $g_j$. If $d = 1$, we are done. Otherwise, there are polynomials $h_1, h_2$ wcd with $g_i = d * h_1$ and $g_j = d * h_2$. For every natural number $m > 0$ we have:

$$\{(x,y) \in I\!R^2 : g_i > 0, g_j > 0\} =$$

$$\{(x,y) \in I\!R^2 : h_1 * h_2 > 0, d * (h_1 + m * h_2) > 0\} \tag{49}$$

This follows from the fact that both sets contain exactly the points where the three polynomials $d, h_1, h_2$ are all positive or all negative.

**Case 1:** $d$ and $h_1 h_2$ are wcd. Every irreducible polynomial can divide $h_1 + m * h_2$ at most for one value of $m$ since otherwise it would divide the difference, hence $h_2$

and $h_1$ too, in contradiction to the assumption that these two polynomials have no common divisor. Consequently, we can choose the value of $m$ such that $h_1+m*h_2$ has no common divisor with any of the polynomials $g_1,\dots,g_n$. In the system $g_1,\dots,g_n$ we replace $g_i$ by $g_i' := d(h_1+m*h_2)$ and $g_j$ by $g_j' := h_1h_2$. We will show that the new system is an equivalent one which is again $(i,j)$-good. So let $(k,l)<(i,j)$. If $k<i$, then it remains to show that $g_k$, $g_i'$ and $g_k$, $g_j'$ have no common divisor. Since $g_i$ and $g_k$ have no common divisor and $d$ is a divisor of $g_i$, it follows that $g_k$ and $d$ are wcd. By choice of $m$ the two polynomials $g_k$ and $h_1+m*h_2$ are without common divisor. Hence $g_k$ and $g_i'$ have no common divisor. Since $g_k$ has no common divisor with $g_i$ nor $g_j$, and $h_1,h_2$ are divisors of these polynomials, $g_j' = h_1h_2$ are wcd too.

If on the contrary $k=i$ and $i<l<j$, then $g_i$ and $g_l$ are wcd, hence $d$ and $g_l$ too. By choice of $m$ $g_l$ and $h_1+m*h_2$ are wcd, hence $g_k$ and $g_i'$ too. So it is shown that the new system is again $(i,j)$-good. But in addition the polynomials $g_i'$ and $g_j'$ have no common divisors, this shows the statement $(*)$ for this case.

**Case 2:** Let $d'$ denote the greatest common divisor of $d$ and $h_1h_2$, so we assume that the degree of $d'$ is at least 1 (otherwise we are in case 1). Then there are polynomials $h_1'$ and $h_2'$ wcd such that $d(h_1+m*h_2) = d'*h_1'$ and $h_1h_2 = d'*h_2'$.

Again, for every natural number $m>0$ we have:

$$\begin{aligned}
&\{(x,y)\in I\!R^2 : h_1'(x,y)*h_2'(x,y)>0, d'(x,y)*(h_1'(x,y)+m*h_2'(x,y))>0\}\\
&=\{(x,y)\in I\!R^2 : h_1(x,y)*h_2(x,y)>0, d(x,y)*(h_1(x,y)+m*h_2(x,y))>0\}\\
&=\{(x,y)\in I\!R^2 : g_i(x,y)>0, g_j(x,y)>0\} \qquad (50)
\end{aligned}$$

Now we choose $m$ such that $h_1'+mh_2'$ has no common divisor with any of the polynomials $g_1,\dots,g_n$. We put $g_i' := d'(h_1'+m*h_2')$ and $g_j' := h_1'h_2'$. Consequently, the system $g_1,\dots,g_{i-1},g_i',g_{i+1},\dots,g_{j-1},g_j',g_{j+1},\dots,g_n$ is equivalent to the system $g_1,\dots,g_n$.

We show firstly that $h_1'h_2'$ and $d'$ have no common divisor. If this were not the case, then let $d''$ be an irreducible divisor of the greatest common divisor of $h_1'h_2'$ and $d'$. Since $h_1'$ and $h_2'$ are wcd, $d''$ must divide one of these two polynomials.

Suppose $d''$ divides $h_2'$. Since $h_1h_2 = d'h_2'$, the polynomial $(d'')^2$ divides $h_1h_2$. Since $h_1$ and $h_2$ have no common divisor, $(d'')^2$ divides one of these two polynomials. But $d''$ divides $d'$ too and consequently $d$, hence $(d'')^3$ divides $g_i$ or $g_j$. This is a contradiction, for we supposed that the polynomials $g_1,\dots,g_n$ contain only first or second powers of irreducible polynomials.

Suppose now that $d''$ divides $h_1'$. Since $d(h_1+m*h_2) = d'h_1'$, the polynomial $(d'')^2$ divides $d(h_1+m*h_2)$. Since $d''$ is also a divisor of $h_1h_2$, the two polynomials $h_1+m*h_2$ and $d''$ cannot have a common divisor. We conclude that $(d'')^2$ divides $d$. But $d''$ also divides one of the polynomials $h_1$ and $h_2$, so $(d'')^3$ divides one of the polynomials $g_i$, $g_j$, this is again a contradiction.

So we have shown that $h_1'h_2'$ and $d'$ have no common divisor. With an analogous proof we can show that the new system is again $(i,j)$-good and that $g_i'$ and $g_j'$ have no common divisor. This shows the statement $(*)$ in this case.

As we have already noted, the proposition follows from $(*)$. $\square$

**Corollary 4.2.3** *For every system of polynomials in $I\!R[x, y]$, there is an equivalent one with a finite number of vertices.*

**Proof:** We apply Proposition 4.2.2. Now the corollary is clear from the fact that two polynomials without common divisor have a finite number of common zeros. $\square$

## 4.3 Reductions with the help of symmetric polynomials

In this section, we want to give another proof of Theorem 4.1.1 which has the advantage that it can be easily generalized to the case that more than 2 functions have a common zero and to give some of the ideas which we will use in Chapter 5. The strategy is to consider the value set of a given description of a semialgebraic set instead of regarding the set itself. In general, this value set is more complex than the first one, but we can describe the image of our semialgebraic set with the help of easier functions. In order to find a reduction at this level, we will produce local descriptions corresponding to the vertices and glue them together in order to get a global description.

**Definition 4.3.1** *Let $x_1, \ldots, x_n \in R$ where $R$ is a real closed field. We denote by $s_i$ the $i^{th}$* **elementary symmetric polynomial***, that is*

$$s_i = \sum x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} \tag{51}$$

*where $(\epsilon_1, \ldots, \epsilon_n)$ runs over all $n$-tuples with $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ and $\sum_j \epsilon_j = i$. We set $\epsilon_0 = 1$.*

**Proposition 4.3.2** *Let $R$ be a real closed field. Then*

$$\{(x_1, \ldots, x_n) \in R^n : x_1 > 0, \ldots, x_n > 0\}$$

$$= \{(x_1, \ldots, x_n) \in R^n : s_1(x_1, \ldots, x_n) > 0, \ldots, s_n(x_1, \ldots, x_n) > 0\} \tag{52}$$

*and*

$$\{(x_1, \ldots, x_n) \in R^n : x_1 \geq 0, \ldots, x_n \geq 0\}$$

$$= \{(x_1, \ldots, x_n) \in R^n : s_1(x_1, \ldots, x_n) \geq 0, \ldots, s_n(x_1, \ldots, x_n) \geq 0\} \tag{53}$$

**Proof:** Let $(x_1, \ldots, x_n) \in R^n$. If all the $x_i$'s are positiv, then the $s_i$'s are trivially positive. Suppose now that not all the $x_i$'s are positive. If a number $x_i$ is zero, then $s_n$ too, so suppose that all the $x_i$'s are positive or negative. Consider the polynomial

$$f(t) = \prod_{i=1}^{n} (t - x_i) = \sum_{i=1}^{n} (-1)^i * s_i * t^{n-i}$$

All the roots are real and not zero, so we can count the number of strictly positive roots with the help of Descartes' Rule (2.1.11), it is the number of variations of signs in the sequence $1, -s_1, s_2, \ldots, (-1)^n s_n$. If the $s_i$'s were all positive, then this number would be $n$, a contradiction.

Let us prove the second part of the proposition. Suppose that we have $x_1, \ldots, x_n$ such that $s_1 \geq 0, \ldots, s_n \geq 0$. By rearranging, we can assume that $x_1, \ldots, x_m \neq 0$ and $x_{m+1} = \ldots = x_n = 0$ for $1 \leq m \leq n$. Consider the polynomial

$$f(t) = \prod_{i=1}^{m}(t - x_i) = \sum_{i=1}^{m}(-1)^i * s_i * t^{n-i}$$

All the roots are real and not zero, so we can count the number of strictly negative roots with the help of Descartes' Rule (2.1.11), it is the number of variations of signs in the sequence $1, s_1, s_2, \ldots, s_m$, hence it is zero. So we find $x_1 \geq 0, \ldots, x_m \geq 0$ and consequently $x_1 \geq 0, \ldots, x_n \geq 0$.

This shows Proposition 4.3.2. □

Next, we introduce a very useful tool which will be the base of Chapter 5. In the proof of Theorem 4.1.1 we saw that the value set of the functions in the given description plays an important rule. One part of statement $(*)$ can be read as follows: there are only finitely many points $x$ on $V$ such that $(f_1(x), f_2(x), f_3(x)) = (1, 1, 0)$ or $= (0, 1, 1)$ or $= (1, 0, 1)$. There are no points $x$ on $V$ such that $(f_1(x), f_2(x), f_3(x)) = (1, 1, c)$ with $c \neq 0$ etc. Consequently, the following definition seems to be interesting and it turns out later that this is the case. Here we don't need it in full generality.

**Definition 4.3.3** *Let $V \subseteq \mathbb{R}^p$ be a real algebraic variety. Given a semialgebraic set $S$ explicitely with the help of a description as a finite union (see Section 2.4)*

$$S = \cup_i \{f_i = 0, g_{i,1} > 0, \ldots, g_{i,n_i} > 0\} \tag{54}$$

*We consider the application*

$$\Phi : V \mapsto \mathbb{R}^m,$$

$$\Phi(x) = (f_1(x), g_{1,1}(x), \ldots, g_{1,n_1}(x), f_2(x), g_{2,1}(x), \ldots) \tag{55}$$

*where $m := \sum_i (1 + n_i)$. We call this map the* **value map** *of the description, the valueset $\Phi(S)$ the* **valueset of** *$S$ and $\Phi(V)$ the* **valueset of** *$V$.*

**Remark 4.3.4** We stress the fact that all these notions depend on the given description. For a fixed semialgebraic set there exist infinitely many ways of describing it and consequently infinitely many different value maps. In the following we take the point of view that we are given a semialgebraic set by an explicit description.

**Theorem 4.3.5** *Let $V$ be a bounded real affine variety over $I\!R$ and let $f_1, \ldots, f_{n+1} \in I\!R[V]$ with $n+1 \geq 2$. Again, we set*

$$P = \{x \in V : f_1(x) > 0, \ldots, f_{n+1}(x) > 0\} \tag{56}$$

*Assume, that the following two conditions hold:*

- *The point $(0, \ldots, 0)$ is not in the closure of the value set of $P$.*
- *There are only finitely many vertices (which are the points on the border of $P$ where exactly $n$ among the functions $f_i$ vanish).*

*Then there is an equivalent system of only $n$ functions $g_1, \ldots, g_n \in I\!R[V]$ i.e.*

$$\{x \in V : f_1(x) > 0, \ldots, f_{n+1}(x) > 0\}$$

$$= \{x \in V : g_1(x) > 0, \ldots, g_n(x) > 0\} \tag{57}$$

**Remark 4.3.6** The following proof is very technical. The idea is to produce a description of a semialgebraic set by describing it locally and then gluing together all these descriptions.

Let us look at the value set $W$ for the variety $V$ of Theorem 4.3.5. Since $P = \{x \in V : f_1(x) > 0, \ldots, f_{n+1}(x) > 0\}$, $\Phi$ is the application from $V$ to $I\!R^{n+1}$ which associates to a point $x$ the $n+1$-tuple $(t_1, \ldots, t_{n+1}) = (f_1(x), \ldots, f_{n+1}(x))$ and $W = \Phi(V)$. Since $V$ is bounded, $W$ is bounded too. We denote the value set of $P$ by $Q$, consequently $Q = W \cap \{t_1 > 0, \ldots, t_{n+1} > 0\}$. The hypothesis on the finiteness of the number of vertices of $P$ means that the intersections of the closure $Q$ with the coordinate lines have a finite number of points. Suppose we can find polynomials $h_1, \ldots, h_n \in I\!R[t_1, \ldots, t_{n+1}]$ such that

$$\{(t_1, \ldots, t_{n+1}) \in W : h_1(t_1, \ldots, t_{n+1}) > 0, \ldots, h_n(t_1, \ldots, t_{n+1}) > 0\}$$

$$= \{(t_1, \ldots, t_{n+1}) \in W : t_1 > 0, \ldots, t_{n+1} > 0\} \tag{58}$$

Then we have

$$P = \{x \in V : h_1(f_1(x), \ldots, f_{n+1}(x)) > 0, \ldots, h_n(f_1(x), \ldots, f_{n+1}(x)) > 0\} \tag{59}$$

This is easy to see: since $x \in V$ we have

$$(f_1(x), \ldots, f_{n+1}(x)) \in W \tag{60}$$

Hence for any $x \in V$ the inequalities $f_1(x), \ldots, f_{n+1}(x) > 0$ are satisfied if and only if the inequalities $h_1(f_1(x), \ldots, f_{n+1}(x)) > 0, \ldots, h_n(f_1(x), \ldots, f_{n+1}(x)) > 0$ are satisfied. In order to reduce the given system, it will be sufficient to find the $n$ functions $h_1, \ldots, h_n$.

Let $x_1, \ldots, x_m \in W$ denote the intersections of $\overline{Q}$ with the coordinate lines. We consider the functions $s_i(t_1, \ldots, t_{n+1})$ for all $i = 3, \ldots, n+1$. (If $n = 1$ then we don't need these functions and sets like $\{s_3 > 0, \ldots, s_{n+1} > 0\}$ are considered to be $W$). We will show that locally in each point $x_i$ we can find a description of the value set of $P$ with only one supplementary function. Afterwards, we have to glue all these functions together in order to obtain a function $h$ such that $h, s_3, \ldots, s_{n+1}$ describe the value set of $P$.

**Remark 4.3.7** The word *neighbourhood* means in the sequel a neighbourhood in $W$ for the topology induced by $I\!R^{n+1}$ and a notation of the form $\{f > 0\}$ is to be read as $\{x \in W : f(x) > 0\}$

**Proposition 4.3.8 (Local description)** *Let $V$ and $P$ be as in Theorem 4.3.5, $Q$ the value set of $P$ and $x_1, \ldots, x_m$ the finitely many intersections of $\overline{Q}$ with the coordinate lines. Then there is a function $h_1 \in I\!R[t_1, \ldots, t_{n+1}]$ and an open neighbourhood $W_1$ of the set $\{x_1, \ldots, x_m\}$ such that*

$$Q \cap W_1 = \{h_1 > 0, s_3 > 0, \ldots, s_{n+1} > 0\} \cap W_1 \tag{61}$$

*and*

$$\{s_3 \geq 0, \ldots, s_{n+1} \geq 0\} \cap W_1 \cap \{h_1 = 0\} = \{x_1, \ldots, x_m\} \tag{62}$$

**Proof:** We set

$$W_1 := \{t_1 + \ldots + t_{n+1} > 0\} \tag{63}$$

Since in a vertex $x_i$ there are exactly $n$ coordinates zero and one positive coordinate, $W_1$ is indeed an open neighbourhood of the set $\{x_1, \ldots, x_m\}$. We define

$$h_1(t_1, \ldots, t_{n+1}) := s_2(t_1, \ldots, t_{n+1}) \tag{64}$$

Since on $W_1$ the first elementary symmetric polynomial $s_1$ is positive, we deduce the first equation from Proposition 4.3.2.

If $x = (t_1, \ldots, t_{n+1})$ is in the set on the left hand side of the second equation, then all the elementary symmetric polynomials of $t_1, \ldots, t_{n+1}$ are non-negative, hence $t_1, \ldots, t_{n+1} \geq 0$ by Proposition 4.3.2. Since $h_1(x) = 0$ and $x \in W_1$ this means that exactly $n$ among the numbers $t_i$ are zero, so $x$ is a vertex and in the set $\{x_1, \ldots, x_m\}$. The inverse inclusion is obvious. $\Box$

**Proposition 4.3.9 (Global description)** *We take the same situation as in Theorem 4.3.5: $V$ a bounded real variety, $P$ a basic open set and $Q$ the value set of $P$. Suppose we have a neighbourhood $W_1$ and a function $h_1$ as in Proposition 4.3.8. Then there exists a polynomial $h \in I\!R[t_1, \ldots, t_{n+1}]$ such that*

$$Q = \{(t_1, \ldots, t_{n+1}) \in W : h(t_1, \ldots, t_{n+1}) > 0,$$

$$s_3(t_1, \ldots, t_{n+1}) > 0, \ldots, s_{n+1}(t_1, \ldots, t_{n+1}) > 0\} \tag{65}$$

Before proving Proposition 4.3.9, we have to show some lemmas.

**Lemma 4.3.10** *There is a non-constant function* $p_1 \in \mathbb{R}[t_1, \dots, t_{n+1}]$ *such that:*

*a)* $p_1 > 0$ *on* $W$*.*

*b)* $p_1(x_i) > 1$ *for* $i = 1, \dots, m$*.*

*c)* $\{p_1 \geq 1\} \subseteq W_1$

**Proof:** This is a simple application of the Stone–Weierstrass–Theorem. □

**Lemma 4.3.11** *There exists a non-constant function* $p_2 \in \mathbb{R}[t_1, \dots, t_{n+1}]$ *such that:*

*a)* $p_2 \geq 0$ *on* $W$*.*

*b)* $p_2(x_i) = 0$ *for* $i = 1, \dots, m$*.*

*c)* $\{p_2 \leq 1\} \subseteq \{p_1 > 1\}$

**Proof:** We set

$$q_i(t_1, \dots, t_{n+1}) := (t_1 - x_{i,1})^2 + \dots + (t_{n+1} - x_{i,n+1})^2 \tag{66}$$

where $x_i = (x_{i,1}, \dots, x_{i,n+1})$. On the compact set $W - \{p_1 > 1\}$ the function $\prod_{i=1}^m q_i$ is strictly positive, hence we find a real number $c > 0$ such that $p_2 := c * \prod_{i=1}^m q_i > 1$ on this set. Then $p_2$ fulfils the assertions of the lemma. □

Now we consider the closed sets

$$M_1 := (W - \{p_2 < 1\}) \quad \cap \quad \overline{Q} \tag{67}$$

$$M_2 := \overline{(W - \{p_2 < 1\}) \quad \cap \quad \{s_3 > 0, \dots, s_{n+1} > 0\} - Q} \tag{68}$$

**Lemma 4.3.12** $M_1$ *and* $M_2$ *are disjoint closed bounded sets.*

**Proof:** Suppose that there is a $x \in M_1 \cap M_2$. We consider four cases:

a) $x \in Q$. Then also a neighbourhood of $x$ is in $Q$, so $x \notin M_2$.

b) $x$ has $i$ coordinate 0 and $n+1-i$ positive coordinates where $i \leq n-1$. Hence $s_1(x) > 0, s_2(x) > 0$ This holds true in a neighbourhood of $x$. Suppose there is a point $y \in (W - \{p_2 < 1\}) \quad \cap \quad \{s_3 > 0, \dots, s_{n+1} > 0\} - Q$ in this neighbourhood. We have consequently $s_1(y) > 0, \dots, s_{n+1}(y) > 0 \Rightarrow y \in Q$ by Proposition 4.3.2. This is a contradiction to $x \in M_2$.

c) $x$ has $n$ coordinates 0 and one positive coordinate. Since $x \in \overline{Q}$, we conclude that $x$ is in the set $\{x_1, \ldots, x_m\}$ and hence $p_2(x) = 0$ and $x \in \{p_2 < 1\}$. Since $\{p_2 < 1\}$ is an open set, every point $y$ in a neighbourhood of $x$ is in this set. This contradicts $x \in M_2$.

d) $x = (0, \ldots, 0)$ This is impossible since $(0, \ldots, 0) \notin \overline{Q}$ by assumption. □

By the Stone-Weierstrass-Theorem we find a polynomial $h_2 \in \mathbb{R}[t_1, \ldots, t_{n+1}]$ such that $h_2 > 1$ on $M_1$ and $h_2 < -1$ on $M_2$ (note that $W$ is closed in $\mathbb{R}^{n+1}$, hence a closed set of $W$ is closed in $\mathbb{R}^{n+1}$ too). Now we set

$$h := p_1^M * h_1 + p_2^M * h_2 \tag{69}$$

where $M$ is a sufficiently large natural number. We claim that this function fulfils the assertions of Proposition 4.3.9. For showing this fact, we have to consider different parts of $W$. For each such part, we will find an exponent $M$ sufficiently large such that $h$ fulfils the assertions of Proposition 4.3.9 on this area. By taking the maximum of these exponents, we find an exponent for $W$ entirely.

**Lemma 4.3.13** *For $M$ sufficiently large,*

$$Q \cap \{p_1 \leq 1\} = \{h > 0, s_3 > 0, \ldots, s_{n+1} > 0\} \cap \{p_1 \leq 1\} \tag{70}$$

**Proof:** On the compact set $W \cap \{p_1 \leq 1\}$ the function $p_1^M h_1$ is bounded independently of $M$. By construction of $p_2$ we have $\{p_1 \leq 1\} \subseteq \{p_2 > 1\}$. But the set on the left is compact, hence we find a real number $\delta > 0$ such that $p_2 > 1 + \delta$ on the set $\{p_1 \leq 1\}$. Now we can choose $M$ sufficiently large such that $h = p_1^M h_1 + p_2^M h_2$ has the same sign as $h_2$ in every point where $|h_2| > 1$.

Let $x \in Q \cap \{p_1 \leq 1\}$. We have to show $x \in \{h > 0, s_3 > 0, \ldots, s_{n+1} > 0\} \cap \{p_1 \leq 1\}$ but the only non-trivial part is to show $x \in \{h > 0\}$. But $p_2(x) > 1$ hence $x \in (W - \{p_2 < 1\}) \cap \overline{Q} = M_1$. Thus $h_2(x) > 1$ and hence $h(x) > 0$.

Conversely, let $x \in \{h > 0, s_3 > 0, \ldots, s_{n+1} > 0\} \cap \{p_1 \leq 1\}$. We have to show $x \in Q \cap \{p_1 \leq 1\}$. Suppose $x \notin Q$. Then $x \in M_2$, hence $h_2(x) < -1$ and $h(x) < 0$, a contradiction.

From these two directions, we conclude Equation 70. □

**Lemma 4.3.14** *For any $M \geq 0$ we have*

$$Q \cap \{p_1 \geq 1, p_2 \geq 1\} = \{h > 0, s_3 > 0, \ldots, s_{n+1} > 0\} \cap \{p_1 \geq 1, p_2 \geq 1\} \tag{71}$$

**Proof:** Let $x \in Q \cap \{p_1 \geq 1, p_2 \geq 1\}$. It remains to show that $h(x) > 0$. On the one hand $x \in M_1$, hence $h_2(x) > 1$. On the other hand, $x \in Q \cap W_1$, hence $h_1(x) > 0$ (see Proposition 4.3.8). It follows $h(x) = p_1(x)^M * h_1(x) + p_2(x)^M * h_2(x) > 0$.

Now let $x \in \{h > 0, s_3 > 0, \ldots, s_{n+1} > 0\} \cap \{p_1 \geq 1, p_2 \geq 1\}$. We have to show $x \in Q$. Suppose $x \notin Q$. Then $x \in M_2$, hence $h_2(x) < -1$. On the other hand $x \in W_1, x \notin Q, s_3(x) > 0, \ldots, s_{n+1}(x) > 0$ and by Proposition 4.3.8 $h_1(x) \leq 0$. Hence $h(x) = p_1(x)^M * h_1(x) + p_2(x)^M * h_2(x) < 0$. This is a contradiction.

These two directions show Equation 71. □

**Lemma 4.3.15** *There is a $M > 0$ sufficiently large such that,*

$$Q \cap \{p_2 \leq 1\} = \{h > 0, s_3 > 0, \dots, s_{n+1} > 0\} \cap \{p_2 \leq 1\} \tag{72}$$

**Proof:** We recall that we have

$$\{s_3 \geq 0, \dots, s_{n+1} \geq 0\} \cap W_1 \cap \{h_1 = 0\} = \{x_1, \dots, x_m\} \tag{73}$$

We set $D := \{s_3 \geq 0, \dots, s_{n+1} \geq 0\} \cap \{p_2 \leq 1\} (\subseteq W_1)$. This is a closed, hence compact set. Since $p_1 > 1$ on $D$, we find a real number $\delta > 0$ such that $p_1 > 1 + \delta$ on $D$.

On $D$ we have $h_1(x) = 0 \Rightarrow x = \{x_1, \dots, x_m\} \Rightarrow p_2(x) = 0$. We apply Łojasiewicz' Inequality (see 2.4.12). This gives us natural numbers $M_2$ and a real number $c_2 > 0$ such that

$$|p_2^{M_2}| \leq c_2 * |h_1| \tag{74}$$

on the set $D$, with equality only on the set $D \cap \{h_1 = 0\} = \{x_1, \dots, x_m\}$.

The function $h_2$ is bounded on $D$. So there is a $c'$ such that on $D$

$$|p_2^{M_2} h_2| \leq c' * |h_1| \tag{75}$$

We choose $M$ sufficiently large such that $(1 + \delta)^M > c'$. Then on $D$

$$|p_2^M h_2| \leq p_1^M * |h_1| \tag{76}$$

with equality only for the points $x_1, \dots, x_m$. This shows that $h = p_1^M h_1 + p_2^M h_2$ and $h_1$ have the same sign on $D$.

Now, we will prove Equation 72.

Let $x \in Q \cap \{p_2 \leq 1\}$. We have to show $x \in \{h > 0, s_3 > 0, \dots, s_{n+1} > 0\} \cap \{p_2 \leq 1\}$. The only non-trivial part is $h(x) > 0$. But

$$x \in Q \cap W_1 = \{h_1 > 0, s_3 > 0, \dots, s_{n+1} > 0\} \cap W_1$$

hence $h_1(x) > 0$. Since $x \in D$, we have $h(x) > 0$.

Conversely, let $x \in \{h > 0, s_3 > 0, \dots, s_{n+1} > 0\} \cap \{p_2 \leq 1\}$. We have to show that $x \in Q \cap \{p_2 \leq 1\}$. It remains to show that $x \in Q$. Suppose $x \notin Q$. Since $x \in W_1$ and $s_3(x) > 0, \dots, s_{n+1} > 0$ we have $h_1(x) \leq 0$, but since $x \in D$ we conclude $h(x) \leq 0$, a contradiction.

So we have proven Equality 72. $\square$

**Proof of Proposition 4.3.9:** We take $M$ sufficiently large such that Equalities 70 and 72 hold. We have

$$\{p_1 \leq 1\} \cup \{p_1 \geq 1, p_2 \geq 1\} \cup \{p_2 \leq 1\} = W \tag{77}$$

We take the union of Equalities 70,71 and 72. This yields to:

$$Q = \{h > 0, s_3 > 0, \ldots, s_{n+1} > 0\} \tag{78}$$

This finishes the proof of Proposition 4.3.9. □

**Proof of Theorem 4.3.5:** By Proposition 4.3.8 and Proposition 4.3.9 we find a function $h \in I\!R[t_1, \ldots, t_{n+1}]$ such that $Q = \{h > 0, s_3 > 0, \ldots, s_{n+1} > 0\}$. As we have already explained (see equation 4.3), this shows Theorem 4.3.5. □

## 4.4 Some consequences

Although Theorem 4.3.5 is not as general as the Theorem of Bröcker–Scheiderer, it has the advantage of a direct and intuitive proof. It is strong enough to give us some useful corollaries, in particular all the theorems we have already proven in this work.

**Corollary 4.4.1** *Let $R$ be an archimedean, real closed field (e.g. $R = R_{alg}$). Then Theorem 4.3.5 holds true with $R$ instead of $I\!R$.*

**Proof:** We consider $R$ as a subfield of $I\!R$. Instead of looking for a reduction of the set $P_R = \{x \in V_R : f_1(x) > 0, \ldots, f_{n+1}(x) > 0\}$, we consider the $f_i$ as functions in $I\!R[x_1, \ldots, x_n]$ and try to reduce $P_{I\!R} = \{x \in V_{I\!R} : f_1(x) > 0, \ldots, f_{n+1}(x) > 0\}$, where $V_{I\!R}$ means the variety defined by the same equations than $V$ but considered as a variety over $I\!R$. Then we have to make sure that the functions obtained by the reduction (which are a priori functions in $I\!R[V]$) lie again in $R[V]$. So we have to follow the proof of Theorem 4.3.5.

The hypothesis that the point $(0, \ldots, 0)$ lies not in the closure of the value set of $P_R$ implies that the first order formula over $R \subseteq I\!R$,

$$\Psi = \exists \epsilon \forall x \in V_R$$

$$(f_1(x) > 0, \ldots, f_{n+1}(x) > 0 \Longrightarrow f_1(x)^2 + \ldots + f_{n+1}(x)^2 > \epsilon^2) \tag{79}$$

is true over $R$. By Model Completeness (see Proposition 2.4.11), we conclude that $\Psi$ is true over $I\!R$ too, so $(0, \ldots, 0)$ is not in the closure of the value set of $P_{I\!R}$.

Next, we express the finiteness of the number vertices with the help of a first order formula. It is easy to find a formula which translates the fact that a point $y \in V_R$ is a vertex. If $m < \infty$ denotes the number of vertives of $P \subseteq V_R$, then the formula

$$\Psi' := \exists x_1, \ldots, x_m \in V_R \forall y \in V_R (y \text{ is a vertex of } P_R$$

$$\Longrightarrow y = x_1 \text{ or } \ldots \text{ or } y = x_m) \tag{80}$$

is true over $R$, hence by Model Completeness it is true over $I\!R$. So $P_{I\!R}$ has exactly $m < \infty$ vertices, and these are the vertices of $P_R \subseteq V_R \subseteq V_{I\!R}$. Consequently, the hypotheses of Theorem 4.3.5 are verified. But if the functions $f_i$ are functions over $R$ and all the vertices have coordinates in $R$, then all the calculations are over $R$. This follows from the fact, that all the functions $p_1, p_2, h_1, h_2$ are in $R[t_1, \ldots, t_{n+1}]$. (For $p_1$ and $h_2$ this is not automatically true, but since $R$ is dense in $I\!R$, we can assume this by variing a bit the coefficients of these functions). Hence $h \in R[t_1, \ldots, t_{n+1}]$ and we are done. □

**Theorem 4.4.2** *Let $V$ be a real affine variety. Given $n+1 \geq 2$ functions $f_1, \ldots, f_{n+1} \in I\!R[V]$. Again, we set*

$$P = \{x \in V : f_1(x) > 0, \ldots, f_{n+1}(x) > 0\} \tag{81}$$

*We suppose, that the following two conditions hold:*

- *$P$ is bounded.*
- *The point $(0, \ldots, 0)$ is not in the closure of the value set of $P$.*
- *There are only finitely many vertices.*

*Then there is an equivalent system of only $n$ functions $g_1, \ldots, g_n \in I\!R[V]$ i.e.*

$$\{x \in V : f_1(x) > 0, \ldots, f_{n+1}(x) > 0\}$$

$$= \{x \in V : g_1(x) > 0, \ldots, g_n(x) > 0\} \tag{82}$$

**Remark 4.4.3** The difference with Theorem 4.3.5 is that here, we only demand that $P$ is bounded, but not that the real affine variety $V$ is bounded.

**Proof:** We use a stereographic projection in order to compactify the real affine variety $V$. Let $V \subseteq I\!R^m$. We consider the map $\Pi : I\!R^m \mapsto S^m - N$ where $N = (0, \ldots, 0, 1)$ and

$$\Pi(t_1, \ldots, t_m) := (y_1, \ldots, y_{m+1}) =$$

$$= \frac{1}{1 + t_1^2 + \ldots + t_m^2}(t_1, \ldots, t_m, t_1^2 + \ldots, t_m^2) \tag{83}$$

The image of $\Pi$ lies in the real algebraic set

$$S^m = \left\{(y_1, \ldots, y_{m+1}) \in I\!R^{m+1} : y_1^2 + \ldots + y_m^2 + (y_{m+1} - \frac{1}{2})^2 = \frac{1}{4}\right\} \tag{84}$$

The inverse mapping is $\Pi^{-1} : S^m - N \mapsto R^m$ defined by

$$\Pi^{-1}(y_1, \ldots, y_{m+1}) = \frac{1}{1 - y_{m+1}}(y_1, \ldots, y_m) \tag{85}$$

If we have an equation $g(x_1,\dots,x_m)=0$ or an inequation $h(x_1,\dots,x_m)>0$ then we consider the equation

$$\overline{g}(y_1,\dots,y_{m+1}) = (1-y_{m+1})^j g\left(\frac{y_1}{1-y_{m+1}},\dots,\frac{y_m}{1-y_{m+1}}\right) = 0 \tag{86}$$

or the inequation

$$\overline{h}(y_1,\dots,y_{m+1}) = (1-y_{m+1})^j h\left(\frac{y_1}{1-y_{m+1}},\dots,\frac{y_m}{1-y_{m+1}}\right) > 0 \tag{87}$$

where the natural number $j$ is sufficiently large such that $\overline{g}$ is a polynomial. For the inequality $h>0$ we take in addition $j$ pair. In this way, the real affine variety $V$ is transformed into a real affine variety $V'$ of $S^m$ and the set $P$ is transformed into a basic open subset $P'$ described by $n+1$ functions. Since $P$ is bounded, $N$ lies not in the closure of $P'$, the vertices of $P$ correspond exactly to the vertices of $P'$ and consequently the assumptions of Theorem 4.3.5 are satisfied, so we find $n$ polynomials in $m+1$ variables which describe $P'$. Replacing an occurrence $y_i$ by $\frac{x_i}{1+x_1^2+\dots+x_m^2}$ for $i=1,\dots,m$ and $y_{m+1}$ by $\frac{x_1^2+\dots+x_m^2}{1+x_1^2+\dots+x_m^2}$ gives rational functions which describe $P$. Multiplying by a sufficiently large power of the strictly positive function $1+x_1^2+\dots+x_m^2$ gives polynomials which describe $P$. □

**Corollary 4.4.4** *Theorem 4.4.2 gives another proof of Theorem 4.1.1.*

**Proof:** The assumptions of Theorem 4.1.1 imply the assumptions of Theorem 4.4.2. □

## 4.5 A direct proof in the archimedean, one–dimensional case

In this section we give a complete proof of the Bröcker-Scheiderer-Theorem in the archimedean, one-dimensional case.

**Theorem 4.5.1** *Let $V$ be a real affine variety of dimension $1$ over a real closed archimedean field $R$. Then every basic open set can be described with one single inequality.*

The idea of the proof is to use a generic reduction and to apply the methods of the preceding sections to produce a true reduction.

We give, without proof, a theorem proven in 1974 by Bröcker:

**Theorem 4.5.2 (Generic reduction)** *Let $R$ be a real closed field and let $V$ be a real affine variety of dimension $d$. Then for every basic open set there exists a basic open set described by only $d$ inequalities such that the symmetric difference of these two sets is of dimension $< d$.*

**Proof of Theorem 4.5.1:**

In order to proof Theorem 4.5.1, it will be sufficient to reduce a basic open set of the form $\{f_1 > 0, g_1 > 0\}$. Since the following demonstration is very similar to that of Theorem 4.3.5, we will not give every detail.

**Case 1:** $V$ is a bounded variety.
Once again, we look at the value set of $V$, this gives us a semialgebraic set of $R^2$ of dimension 1 and the Zariski–closure $W$ of this set has also dimension 1. The aim is to find a function $f \in R[t_1, t_2]$ such that

$$\{(t_1, t_2) \in W : t_1 > 0, t_2 > 0\} = \{(t_1, t_2) \in W : f(t_1, t_2) > 0\} \tag{88}$$

We apply Theorem 4.5.2 to the real affine variety $W$, hence $d = 1$. This gives us a function $f \in R[W]$ such that equality 88 is satisfied except on a set of dimension 0, that is a finite number of points. We choose a representant of $f$ in $R[t_1, t_2]$ which we call $f$ again. If there are points $(t_1, t_2)$ such that $f(t_1, t_2) > 0$ but $t_1 \leq 0$ or $t_2 \leq 0$ then we multiply $f$ by a function which has a root in $x$ and which is strictly positive elsewhere. Executing this algorithm for all such points, we can suppose that

$$\{(t_1, t_2) \in W : f(t_1, t_2) > 0\}$$

$$= \{(t_1, t_2) \in W : t_1 > 0, t_2 > 0\} - \{x_1, \ldots, x_m\} \tag{89}$$

where $\{x_1, \ldots, x_m\}$ are the (finitely many) points of the difference of the two sets. So all the coordinates of the $\{x_1, \ldots, x_m\}$ are positive.

**Lemma 4.5.3** *There are functions $p_1, p_2 \in R[t_1, t_2]$ which are non-negative on $W$ and such that*

$$\{x_1, \ldots, x_m\} \subseteq \{p_1 < 1\} \subseteq \{p_1 \leq 1\} \subseteq$$

$$\subseteq \{p_2 > 1\} \subseteq \{p_2 \geq 1\} \subseteq \{t_1 > 0, t_2 > 0\} \tag{90}$$

*and*

$$\{t_1 t_2 = 0\} \subseteq \{p_2 = 0\} \tag{91}$$

**Proof (Sketch):** We start with the function $(t_1 t_2)^2$ and multiply by a positive polynomial function which is sufficiently large on the points $\{x_1, \ldots, x_m\}$ and sufficiently small on the set $\{t_1 < 0\} \cup \{t_2 < 0\}$. (Existence by Stone–Weierstrass). This yields to the function $p_2$. Then also by Stone–Weierstrass we find the function $p_1$.
$\square$

The two closed sets $\{p_2 \geq 1\}$ and $\{t_1 \leq 0\} \cup \{t_2 \leq 0\}$ are disjoint, by Stone–Weierstrass we find a function $g$ such that $g > 1$ on the first set and $g < -1$ on the second set.

**Lemma 4.5.4** *Let $M$ be a sufficiently large natural number and set*

$$h := p_1^M f + p_2^M g \tag{92}$$

*Then*

$$\{(t_1, t_2) \in W : h(t_1, t_2) > 0\} = \{(t_1, t_2) \in W : t_1 > 0, t_2 > 0\} \tag{93}$$

**Proof:** This is essentially the same as in the proof of Theorem 4.3.5. On the set $\{p_1 < 1\}$ the function $p_1^M f$ is bounded independently of $M$ whereas $p_2 > 1 + \epsilon$ for a $\epsilon > 0$. Consequently, for $M$ sufficiently large, $h > 0$ and hence Equation 93 is satisfied on $\{p_1 < 1\}$.

On the set $\{p_2 \geq 1, p_1 \geq 1\}$ we have $f > 0, g > 0$ and hence $h > 0$ and Equation 93 is verified on this set.

On the set $\{t_1 \leq 0\} \cup \{t_2 \leq 0\}$ we have $f \leq 0, g \leq 0$ hence $h \leq 0$ and Equation 93 is verified on this set.

Now, we consider the closed set $\{t_1 \geq 0, t_2 \geq 0, p_2 \leq 1\}$. By assumption, the only roots of $f$ which lie in this set must have a vanishing coordinate and hence $p_2 = 0$ on these points. This enables us to apply Łojasiewicz' Inequality and with the same arguments as in the proof of Lemma 4.3.15, for $M$ sufficiently large, Equation 93 is verified on this set. □

From the lemma above we conclude that

$$\{x \in V : f_1(x) > 0, f_2(x) > 0\} = \{x \in V : h(f_1(x), f_2(x)) > 0\} \tag{94}$$

which shows Theorem 4.5.1 in the bounded case.

**Case 2:** $V$ is not bounded.
With the help of a stereographic projection, this case can be reduced to the first one. Since this is exactly the same as in the proof of Theorem 4.4.2, we omit the details. □

# 5 Polynomial Reductions

## 5.1 Polynomial Reduction of basic open sets

All the constructions in Chapter 4 have a common point. Starting with any description of a semialgebraic, basic open set we found another, equivalent description for it, for which the used functions depended in a *polynomial way* on the given functions. The next definition gives a precise meaning to this notion.

**Definition 5.1.1** *Let $V$ be a real variety over an ordered field $R$ and let $S \subseteq V$ be a basic open set written in the form*

$$S = \{x \in V : f_1(x) > 0, \ldots, f_m(x) > 0\} \tag{95}$$

*where the $f_1, \ldots, f_m \in R[V]$. A* **polynomial reduction with $s < m$ functions** *is a set of $s$ functions $h_1, \ldots, h_s \in R[t_1, \ldots, t_m]$ such that*

$$S = \{x \in V : h_1(f_1(x), \ldots, f_m(x)), \ldots, h_s(f_1(x), \ldots, f_m(x))\} \tag{96}$$

**Remark 5.1.2** This definition depends on an explicit description of $S$ as a basic open set.

The notion *polynomial reduction* can apply to more general cases:

**Definition 5.1.3** *Let $V$ be a real variety over a field $R$ and $S \subseteq V$ a semialgebraic set which is explicitely given by a description with the help of $m$ functions $f_i \in R[V]$. Then another description of $S$ with the help of functions $g_j$ is called a* **polynomial description** *if for every $g_j$ there exists a polynomial $h_j \in R[t_1, \ldots, t_m]$ such that for any $x \in V$*

$$g_j(x) = h_j(f_1(x), \ldots, f_m(x)) \tag{97}$$

We want to prove the following theorem:

**Theorem 5.1.4 (Basic open sets)**
*Let $V$ be a real variety of dimension $d$ over a real closed field $R$ and $S \subseteq V$ a semialgebraic basic open set given by an explicite description. Then there exists a polynomial reduction with $d$ functions.*

**Proof:** It is sufficient to prove that we can reduce in a polynomial way every set of $d+1$ functions, since by induction we then find polynomial reductions for every set of functions. We use the function $\Phi$ introduced in Definition 4.3.3. We recall that $\Phi : V \mapsto R^{d+1}$. Let $W$ be the image of $V$ under this mapping. By Proposition 2.6.3 $\dim W \leq \dim V = d$. By Proposition 2.4.10, $W$ is a semialgebraic subset of $R^{d+1}$, let $W'$ denote the Zariski–closure of $W$. Then the dimension of $W'$ is at most $d$ (see Proposition 2.6.2).

We consider the semialgebraic set

$$\{y = (t_1, \ldots, t_{d+1}) \in W' : t_1 > 0, \ldots, t_{d+1} > 0\} \tag{98}$$

We consider the functions $t_1, \ldots, t_{d+1}$ as elements of $R[W']$. By Theorem 2.7.1 we find $h_1, \ldots, h_d \in R[W']$ such that

$$\{y \in W' : t_1 > 0, \ldots, t_{d+1} > 0\}$$

$$= \{y \in W' : h_1(t_1, \ldots, t_{d+1}) > 0, \ldots, h_1(t_1, \ldots, t_{d+1}) > 0\} \tag{99}$$

Since $R[W'] = R[t_1, \ldots, t_{d+1}]/I(W')$, we can choose for each $h_i$ a representant in $R[t_1, \ldots, t_{d+1}]$ which, by abuse of notation, we call $h_i$ again.

Now we have

$$\{x \in V : f_1(x) > 0, \ldots, f_{d+1}(x) > 0\}$$

$$= \{x \in V : h_1(f_1(x), \ldots, f_{d+1}(x)) > 0, \ldots, h_d(f_1(x), \ldots, f_{d+1}(x)) > 0\} \tag{100}$$

This is easy to see: if $x \in V$ then $(f_1(x), \ldots, f_{d+1}(x)) \in W \subseteq W'$, hence by Equation 99 the Equation 100 follows immediately. $\Box$

**Problem 5.1.5** There are some natural questions. For instance, we could ask if we are able to bound the degrees of the polynomials $h_1, \ldots, h_d$ by a bound which depends only on $V$. Another problem would be to bound the degrees of $h_1, \ldots, h_d$ by a bound which depends only on $V$ and the maximal degree of the polynomials $f_1, \ldots, f_{d+1}$. These questions seems not to be very easy.

## 5.2 Polynomial Reduction of semialgebraic sets

In this section, we will follow the same strategy as in the previous to find polynomial reductions for semialgebraic sets.

**Theorem 5.2.1 (Semialgebraic sets)**
*Let $V$ be a real variety $V$ of dimension $d$ and $S \subseteq V$ a semialgebraic set given explicitely by a description which uses the functions $f_1, \ldots, f_m \in R[V]$ Then there is a polynomial reduction of $S$ as*

$$S = \bigcup_{i=1}^{t} \{g_i = 0, g_{i,1} > 0, \ldots, g_{i,s} > 0\} \tag{101}$$

*with $t \leq \tau(d)$ and $s \leq d$ where the $g_i, g_{i,j} \in R[V]$ depend in a polynomial way on the $f_1, \ldots, f_m$.*

**Proof:** Again, we consider the value map $\Phi : V \mapsto R^m$ defined by

$$\Phi(x) = (f_1(x), \ldots, f_m(x)) \tag{102}$$

By Propositions 2.6.3 the dimension of the semialgebraic set $W = \Phi(V)$ is at most $d$. Then with Proposition 2.6.2 the dimension of the Zariski–Closure of $W$, which we denote by $W'$, is at most $d$. Now we use Theorem 2.7.5. We consider the real algebraic variety $W'$ and the semialgebraic set $P \subseteq W'$ which arises from the description of $S$ by replacing every occurence of the form $f_i(x) > 0$ (resp. $\geq 0$, $< 0$, $\leq 0$) by $t_i > 0$ (resp. $\geq 0$, $< 0$, $\leq 0$). So we find a description of $P$ of the form

$$P = \bigcup_{i=1}^{t} \{(t_1, \ldots, t_m) \in W' : h_i = 0, h_{i,1} > 0, \ldots, h_{i,s} > 0\} \tag{103}$$

with $h_i, h_{i,j} \in R[W']$, $t \leq \tau(d)$ and $s \leq d$. Again we denote by $h_i$ resp. $h_{i,j}$ a representant of $h_i$ resp. $h_{i,j}$ in $R[t_1, \ldots, t_m]$. But then with $f(x) := (f_1(x), \ldots, f_m(x))$

$$S = \bigcup_{i=1}^{t} \{x \in V : h_i(f(x)) = 0, h_{i,1}(f(x)) > 0, \ldots, h_{i,s}(f(x)) > 0\} \tag{104}$$

This is easy to see, since if $x \in S$, then $(f_1(x), \ldots, f_m(x)) \in P$. This shows Theorem 5.2.1. $\square$

## 5.3 Polynomial Reduction of other classes of sets

With the tool developped in the preceding sections, we can prove some other theorems for polynomial reductions. We have to use the corresponding theorems in Section 2.7.

**Theorem 5.3.1 (Basic closed sets)**
*Let $V$ be a real variety of dimension $d$ over a real closed field $R$ and $S \subseteq V$ a semialgebraic basic closed set given by an explicite description. Then there exists a polynomial reduction of the form*

$$S = \{h_1(x) \geq 0, \ldots, h_{\overline{s}}(x) \geq 0\} \tag{105}$$

*with $h_i \in R[V]$ and $\overline{s} \leq \frac{d(d+1}{2}$*

**Theorem 5.3.2 (Open semialgebraic sets)**
*Let $V$ be a real variety of dimension $d$ over a real closed field $R$ and $S \subseteq V$ a open semialgebraic set given by an explicite description*

$$S = \bigcup_{i=1}^{u} \{f_{i,1} > 0, \ldots, f_{i,w_i} > 0\} \tag{106}$$

*Then there exists a polynomial reduction of the form*

$$S = \bigcup_{i=1}^{t} \{h_{i,1}(x) > 0, \ldots, h_{i,s}(x) > 0\} \tag{107}$$

*with* $h_{i,j} \in R[V]$, $s \leq d$ *and* $t \leq (d+1) * \tau(d)$

**Problem 5.3.3** Can we achieve $t \leq t(V)$ in general? Proposition 2.6.3 states that the dimension of the image of a semialgebraic set of dimension $d$ under a semialgebraic function has a dimension $\leq d$, but it is not obvious (and probably not true) why the $t$-invariant of the image cannot be greater than the one of the considered semialgebraic set.

**Theorem 5.3.4 (Closed semialgebraic sets)**
*Let* $V$ *be a real variety of dimension* $d$ *over a real closed field* $R$ *and* $S \subseteq V$ *a closed semialgebraic set given by an explicite description*

$$S = \bigcup_{i=1}^{u} \{f_{i,1} \geq 0, \ldots, f_{i,w_i} \geq 0\} \tag{108}$$

*Then there exists a polynomial reduction of the form*

$$S = \bigcup_{i=1}^{\overline{t}} \{h_{i,1}(x) \geq 0, \ldots, h_{i,\overline{s}}(x) \geq 0\} \tag{109}$$

*with* $h_{i,j} \in R[V]$, $\overline{s} \leq \frac{d(d+1)}{2}$ *and* $\overline{t} \leq d^{(d+1)*\tau(d)}$

**Proofs:** The proofs are similar to the proofs of Theorems 5.1.4 and 5.2.1. Given an explicit description of the semialgebraic set with functions $f_1, \ldots, f_m$, we consider the image of $S$ under the application $\Phi$ which associates to $x \in V$ the point $(f_1(x), \ldots, f_m(x)) \in R^m$. We consider the Zariski–closure $W'$ of the image of $\Phi$. By Propositions 2.6.3 and 2.6.2, its dimension is $\leq d$ if $d$ denotes the dimension of $V$.

In the description of $S$ we replace every occurence of the form $f_i(x) > 0$ (resp. $\geq 0$ etc.) by $t_i > 0$ (resp. $t_i \geq 0$ etc.). It is clear from the descriptions of $S$ that this gives us a basic closed resp. an open semialgebraic resp. a closed semialgebraic subset of $W'$. Now, we apply Theorems 2.7.6 resp. 2.7.4 resp. 2.7.7 to find reductions of these semialgebraic sets. Replacing $t_i$ by $f_i(x)$ in these descriptions gives us the polynomial reductions of $S$ we sought in the theorems. □

## 5.4 A spectral version

An analysis of the methods in the previous sections allows us to generalize the results to a wide class of objects. While we used so far arguments from Semialgebraic Geometry, we next want to find a spectral version of the theorems about polynomial reductions.

**Theorem 5.4.1** *Let $A$ be a $R$-algebra of finite transcendence degree $d$. Then every basic open set of* $\mathrm{Spec_r} A$ *can be written with at most $d$ inequalities.*

**Proof:** It is sufficient to show this theorem for a basic open set described with $d+1$ inequalities. So let $P = \{\alpha \in \mathrm{Spec_r} A : a_1(\alpha) > 0, \ldots, a_{d+1}(\alpha) > 0\}$. Since the transcendence degree of $A$ is $d$, we find a non zero polynomial $p \in R[t_1, \ldots, t_{d+1}]$ such that $p(a_1, \ldots, a_{d+1}) = 0$. We set

$$B := R[t_1, \ldots, t_{d+1}]/(p) \tag{110}$$

Consider the application

$$\phi : B \mapsto A \tag{111}$$

defined by $\overline{t_i} \mapsto a_i$. It is clear that $\phi$ is well defined. It induces a map

$$\phi^* : \mathrm{Spec_r} A \mapsto \mathrm{Spec_r} B \tag{112}$$

Set

$$Q = \{\beta \in \mathrm{Spec_r} B : t_1(\beta) > 0, \ldots, t_{d+1}(\beta) > 0\} \tag{113}$$

Then $P = \phi^{*-1}(Q)$:

$$\alpha \in P = \{a_1 > 0, \ldots, a_{d+1} > 0\} \iff \forall i \ -a_i \notin \alpha \iff \forall i \ -\phi(t_i) \notin \alpha$$

$$\iff \forall i \ -t_i \notin \phi^*(\alpha) \iff \phi^*(\alpha) \in Q = \{t_1 > 0, \ldots, t_{d+1} > 0\} \tag{114}$$

Since $p$ is a non-zero polynomial, the transcendence degree of $B$ is at most $d$. By Theorem 2.7.3 we can write $Q$ with only $d$ functions $\overline{g_1}, \ldots, \overline{g_d} \in B$ such that

$$Q = \{\beta \in \mathrm{Spec_r} B : \overline{g_1}(\beta) > 0, \ldots, \overline{g_d}(\beta) > 0\} \tag{115}$$

We choose for each $\overline{g_i}$ a representant $g_i \in R[t_1, \ldots, t_{d+1}]$ We claim that

$$P = \{\alpha \in \mathrm{Spec_r} A : g_1(a_1, \ldots, a_{d+1}) > 0, \ldots, g_d(a_1, \ldots, a_{d+1}) > 0)\} \tag{116}$$

For the proof, let $\alpha \in P$. Then

$$\forall i \ g_i(a_1, \ldots, a_{d+1}) \notin \alpha \iff \forall i \ \phi(\overline{g_i}(t_1, \ldots, t_{d+1})) \notin \alpha$$

$$\iff \forall i \ \overline{g_i}(t_1, \ldots, t_{d+1}) \notin \phi^*(\alpha) \tag{117}$$

Consequently,

$$\alpha \in P \iff \phi^*(\alpha) \in Q \iff \phi^*(\alpha) \in \{\overline{g_1} > 0, \ldots, \overline{g_d} > 0\}$$

$$\iff \alpha \in \{g_1(a_1, \ldots, a_{d+1}) > 0, \ldots, g_d(a_1, \ldots, a_{d+1}) > 0\} \tag{118}$$

This shows the claim and Theorem 5.4.1. $\square$

## References


[1] Knebusch M.; Scheiderer C.: Einführung in die reelle Algebra.- Braunschweig: Vieweg, 1989.- 184 S.

[2] Bochnak, J.; Coste, M.; Roy, M.–F.: Géométrie Algébrique Réelle. Ergebnisse der Mathematik und ihrer Grenzgebiete (3. Folge) 12.- Berlin: Springer, 1987

[3] Andradas C., Bröcker L., Ruiz J.M.: Constructible Sets in Real Geometry.- Berlin: Springer-Verlag, 1996.-270 S.

[4] Marshall, M.: Minimal generation of basic sets in the real spectrum of a commutative ring.- in: Recent advances in real algebraic geometry and quadratic forms.- Berkeley 1990/91.- S.207-219

[5] Marshall, M.: Minimal generation of constructible sets in the real spectrum of a ring.- in: Matematica–Contemporanea 6(1994).- S. 61-128

[6] Marshall, M.: Minimal generation of basic semi-algebraic sets over an arbitrary ordered field.- in: Real algebraic geometry, Rennes 1991.- S.346-353

[7] Prestel, A.: Einführung in die Mathematische Logik und Modelltheorie.- Braunschweig: Vieweg, 1986.- 286 S.

[8] Coste, M.: Introduction à la géométrie semi-algébrique, lectures 1993 and 1997

[9] vom Hofe, G.: Beschreibung von ebenen konvexen n–Ecken durch höchstens drei algebraische Ungleichungen.- Dissertation, Dortmund, 1992

[10] Łojasiewicz, S.: Sur le problème de la division. in: Studia Math. 18, 1959.- S. 87-136

[11] Precup, R.: Estimates of the degree of comonotone interpolating polynomials.- in: L'analyse numérique et la théorie de l'approximation, tome 11, N.1-2, 1982.- S. 139-145

[12] Precup, R.: Piecewise convex interpolation.- in: L'analyse numérique et la théorie de l'approximation, tome 14, N.2, 1985.- S. 123-126

[13] Bröcker, L.: On the separation of basic semialgebraic sets by polynomials.- in: Manuscripta math. 60, 1988.- S. 497-508

[14] Mahé, L.: Une démonstration élémentaire du théorème de Bröcker-Scheiderer.- in: Comptes Rendus de l'Academie des Sciences, Serie I, Mathematique 309(1989).- S. 613- 616

[15] Burési, J.; Mahé L.: Reducing inequalities with bounds.- Prépublication 95-36, Université de Rennes I

[16] Pfister, A.: Zur Darstellung definiter Funktionen als Summe von Quadraten.- in: Invent. math. 4, 1967.- S. 229-237

[17] Hartshorne, R.: Algebraic Geometrie.- Graduate Texts in Math 52.- Berlin: Springer, 1977