CURVATURE BOUNDS ON SUBANALYTIC SPACES

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Abstract. Curvature bounds which play the role of Ricci, scalar curvature and Einstein tensor bounds are introduced for subanalytic topological manifolds. It is shown, using metric properties of subanalytic sets, that an upper (lower) bound on the sectional curvature in the sense of Alexandrov implies an upper (lower) bound on the Ricci curvature and on the Einstein tensor. In the same way, an upper (lower) bound on the Ricci curvature or on the Einstein tensor implies an upper (lower) bound on the scalar curvature.

1. Introduction

We recently showed that there is no reasonable notion of Ricci tensor for subanalytic spaces, whereas one can define an Einstein tensor for such spaces ([5]). Using some heuristics, one can still guess, by looking at the Einstein tensor, what Ricci curvature bounds on subanalytic spaces could look like. The aim of this paper is to define and study such curvature bounds. We construct a complete theory of curvature bounds for subanalytic spaces including sectional curvature, Ricci curvature, Einstein tensor and scalar curvature. The main theorem states roughly that all classical implications between such curvature bounds also hold in the subanalytic context. It is a generalization of the theorems in [1] and [2], the proof presented here being easier thanks to the use of quasi-geodesics.

Let M be a real analytic manifold endowed with a Riemannian metric g. Consider a compact subanalytic subset $X \subset M$. For the definition of subanalytic sets, see 3.1. We suppose that X is a connected topological manifold of dimension n. The metric g of the ambient space M induces a length metric d on X. Any two points in X can be connected by a (not necessarily unique) geodesic, i.e. (X, d) is a geodesic metric space.

We define curvature bounds which will play the same role as Ricci, scalar curvature and Einstein tensor bounds on Riemannian manifolds. For the case of sectional curvature, we can use the existing theory of Alexandrov spaces with lower or upper curvature bounds.

For Ricci and scalar curvature and the Einstein tensor, we use a Verdier stratification of X (see 3.2). Strata of highest dimension are just Riemannian

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manifolds and therefore have a Ricci tensor, an Einstein tensor and a scalar curvature. A neighborhood of a point x of a stratum S of codimension 1 (i.e. dimension $n-1$) is the union of S with two C^1-manifolds Γ_1, Γ_2 with boundary S. The tangent space of $\Gamma_i, i = 1, 2$ at x splits as $T_xS \oplus \mathbb{R}v_+^i$, where v_+^i is a unit vector pointing to Γ_i.

We define the second fundamental form of S in X at the point x by setting $II(u,u) := II_{v_+^1}(u,u) + II_{v_+^2}(u,u)$. In other words, we consider S as boundary of Γ_1 and Γ_2, take the corresponding second fundamental forms and add them.

We also need the density of X at a point $x \in X$. This is the limit $\theta(X,x) := \lim_{r\to 0+} \frac{\mathcal{H}^n(X\cap B(x,r))}{b_n r^n}$, where $\mathcal{H}^n$ denotes n-dimensional Hausdorff volume and b_n is the volume of the unit ball in $\mathbb{R}^n$. The limit exists by a theorem of Kurdyka-Raby ([19]). Instead of taking the ball $B(x,r)$ of the ambient space, we could as well take the ball $B_i(x,r)$ for the induced length metric on X. The corresponding density is the same by an easy argument in [1].

The main definition is the following.

Definition 1.1. Curvature bounds on subanalytic spaces
Let X be a compact subanalytic subset of a real analytic manifold M, endowed with a Riemannian metric g. Suppose that X is a topological manifold. Let κ be a real number.

- *X has sectional curvature bounded from above (below) by κ, if (X,d) is a metric space with curvature bounded from above (below) by κ in the sense of Alexandrov (see Section 2 for details).*
- *X has Ricci curvature bounded from above (below) by $(n-1)\kappa$, if for one Verdier stratification (and then for all such stratifications) the following three conditions hold:*
  - *On a highest dimensional stratum S, $ric \leq (n-1)\kappa g|_S$ ($ric \geq (n-1)\kappa g|_S$), where ric is the $(0,2)$ Ricci tensor of S.*
  - *On a stratum S of codimension 1, $II \leq 0$ ($II \geq 0$).*
  - *For all points $x \in X$, $\theta(X,x) \geq 1$ ($\theta(X,x) \leq 1$).*
- *X has Einstein tensor bounded from above (below) by $\binom{n-1}{2}\kappa$, if for one Verdier stratifications (and then for all such stratifications) the following three conditions hold:*
  - *On a highest dimensional stratum S, $E := \frac{s}{2}g - \mathrm{ric} \leq \binom{n-1}{2}\kappa g|_S$ ($E \geq \binom{n-1}{2}\kappa g|_S$).*
  - *On a stratum S of codimension 1, $\mathrm{tr}\, II g|_S - II \leq 0$ ($\mathrm{tr}\, II g|_S - II \geq 0$).*
  - *For all points $x \in X$, $\theta(X,x) \geq 1$ ($\theta(X,x) \leq 1$).*
- *X has scalar curvature bounded from above (below) by $n(n-1)\kappa$, if for one Verdier stratification (and then for all such stratifications) the following three conditions hold:*
  - *On a highest dimensional stratum S, $s \leq n(n-1)\kappa$ ($s \geq n(n-1)\kappa$), where s is the scalar curvature of S.*
  - *On a stratum S of codimension 1, $\mathrm{tr}\, II \leq 0$ ($\mathrm{tr}\, II \geq 0$).*
  - *For all points $x \in X$, $\theta(X,x) \geq 1$ ($\theta(X,x) \leq 1$).*

As a first example, note that these curvature bounds reduce to the classical bounds if X is a smooth submanifold of M. What is more interesting is that our definition moreover takes care of the singularities of X. If X is twodimensional and subanalytic, then sectional, Ricci and scalar curvature bounds all agree, as in Riemannian geometry. However, this is not so easy to see. It follows from the results of [10].

The simplest relation between curvatures in Riemannian geometry is the fact that bounds on the sectional curvature imply bounds on Ricci curvature and Einstein tensor, whereas bounds on Ricci curvature as well as bounds on Einstein tensor imply bounds on scalar curvature. The question arises if this is also the case in the subanalytic setting. The (positive) answer is the main result in this paper.

Theorem 1. Comparison theorem
Let X be a compact subanalytic subset of a real analytic manifold M, equipped with a Riemannian metric g. Suppose that X is a topological manifold of dimension n.

- *If X has sectional curvature bounded from above (below) by κ, then its Ricci curvature is bounded from above (below) by $(n-1)\kappa$ and its Einstein tensor is bounded from above (below) by $\binom{n-1}{2}\kappa$.*
- *If X has Ricci curvature bounded from above (below) by $(n-1)\kappa$, or if X is of dimension ≥ 3 and has Einstein tensor bounded from above (below) by $\binom{n-1}{2}\kappa$, then its scalar curvature is bounded from above (below) by $n(n-1)\kappa$.*

The obvious corollary that a bound κ on the sectional curvature implies a bound $n(n-1)\kappa$ on the scalar curvature was proved in [1] and [2] (with a slightly weaker notion of bounds for scalar curvature). In comparison to the cited papers, the proof presented here is easier in some respects. First of all, a better understanding of the density in metric spaces is obtained by using the (easy) Propositions 2.4 and 2.5. Secondly, the case of strata of codimension 1 is simplified by using Proposition 4.1. This proposition is the main technical argument in the proof of the comparison theorem, since it relates the metric geometry of X near a stratum of codimension 1 to the second fundamental form of the stratum. For the proof, we have to use all we know about subanalytic spaces. In particular, Pawłucki's theorem gives a better insight in the local structure near such a stratum, the Verdier and Whitney conditions are used several times and estimates on the length of geodesics are proved. Finally, the theory of quasi-geodesics is applied in combination with Proposition 4.1 to give a shorter and more general proof in the case of lower curvature bounds.

Our main theorem is a satisfactory generalization of the classical implications between curvature bounds and shows that the definition of Ricci curvature bounds is the right choice. However, the generalization of the classical Bochner and Myer theorems would require a more detailed study of geodesics on subanalytic spaces, including a second variation formula.

Let us mention that there exists a theory of collapsing for lower Ricci curvature bounds by Cheeger-Colding ([13]). In particular, they also define (lower) Ricci curvature bounds on metric spaces, but new ideas would be necessary to compare with our bounds. The reason is that one knows too less about volumes of (metric) balls on subanalytic sets.

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2. Alexandrov spaces with curvature bounded from above or from below

2.1. **Geodesic spaces and curvature bounds.** We can not give here, of course, a satisfactory treatment of spaces with curvature bounds. We refer the reader to [11], [9] and [12] for much more details and proofs. However, we will include some basic notions and properties in order to make this paper as much self-contained as possible.

In a metric space (X, d), a geodesic is the image of an isometry $[0, L] \to X, L \geq 0$. Equivalently, a geodesic between $x, y \in X$ is a path of length $d(x, y)$ between x and y. Similarly, a local geodesic is the image of a local isometry $[0, L] \to X, L \geq 0$.

A geodesic metric space is by definition a metric space in which every pair of points (x, y) can be connected by a geodesic. If every pair of points (x, y) with $d(x, y) < D$ can be connected by a geodesic, then (X, d) is called D geodesic.

A geodesic triangle in (X, d) consist of three points $x, y, z \in X$ together with three geodesics $[x, y], [x, z], [y, z]$. The perimeter of such a triangle is the sum $d(x, y) + d(y, z) + d(z, x)$.

Let M^2_κ be the unique complete, simply connected two-dimensional Riemannian manifold of constant curvature κ and D_κ its diameter, i.e. $D_\kappa = \infty$ if $\kappa \leq 0$ and $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$. If the diameter of a geodesic triangle is less than $2D_\kappa$, then there exists a comparison triangle $\tilde{x}, \tilde{y}, \tilde{z}$ in M^2_κ such that $d(\tilde{x}, \tilde{y}) = d(x, y), d(\tilde{x}, \tilde{z}) = d(x, z), d(\tilde{y}, \tilde{z}) = d(y, z)$. Given $p \in [x, y], q \in [x, z]$, there are uniquely defined comparison points $\tilde{p} \in [\tilde{x}, \tilde{y}], \tilde{q} \in [\tilde{x}, \tilde{z}]$ such that $d(\tilde{x}, \tilde{p}) = d(x, p), d(\tilde{x}, \tilde{q}) = d(x, q)$.

Definition 2.1.

- *A metric space is a CAT(κ)-space if it is D_κ geodesic and for all geodesic triangles $(x, y, z, [x, y], [x, z], [y, z])$ of perimeter less than $2D_\kappa$ and all points $p \in [x, y], q \in [x, z]$*

$$d(\tilde{p}, \tilde{q}) \geq d(p, q).$$

- *X is said to have curvature bounded from above by κ if for each $x \in X$, there exists $r > 0$ such that $B(x, r) = \{y \in X, d(y, x) \leq r\}$ with the induced metric is a $CAT(\kappa)$ space.*
- *X is said to have curvature bounded from below by κ, if it is geodesic and each point has a neighborhood U of diameter less than D_κ such*

that for all geodesic triangles $(x, y, z, [x, y], [x, z], [y, z])$ *contained in* U *and all points* $p \in [x, y], q \in [x, z]$

$$d(\tilde{p}, \tilde{q}) \leq d(p, q).$$

In intuitive terms, small triangles in a space of curvature bounded from above (below) are "thinner" ("fatter") than triangles in M^2_κ.

A first difference between these two kinds of curvature bounds is that in the case of lower curvature bounds, one has a globalization theorem which states that the triangle comparison holds even in the large. This theorem is also known as Toponogov's theorem.

In a space X of curvature bounded from above, local geodesics can be extended under weak topological assumptions. If for each point $x \in X$ there exists $r > 0$ such that the set $B(x, r) \setminus \{x\}$ is not contractible, then local geodesics can be extended indefinitely. The assumption holds, for instance, if X is a topological manifold or a homology manifold. We say in this case that X has the geodesic extension property. We note further that local geodesics of length at most D_κ in a CAT(κ)-space are geodesics.

2.2. **Densities in spaces with curvature bounds.** For later use, we will prove some propositions concerning densities in spaces with (upper or lower) curvature bounds.

Proposition and Definition 2.2. *Let* X *be a space of curvature bounded from above by* $\kappa \in \mathbb{R}$ *which has the geodesic extension property. Then for all* $x \in X$ *and all real* $\alpha \geq 0$*, the limit*

$$\theta_\alpha(X, x) := \lim_{r \to 0} \frac{\mathcal{H}^\alpha(B(x, r))}{b_\alpha r^\alpha} \in [0, +\infty]$$

exists, where $b_\alpha = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2}+1)}$ *is the volume of the "*α*-dimensional unit ball" and* $\mathcal{H}^\alpha$ *denotes* α*-dimensional Hausdorff measure. This limit is called the* α*-dimensional density of* X *at* x*.*

Proof: The curvature bound implies that there is some R_0 such that the ball $B(x, R_0)$, endowed with the induced metric, is a CAT(κ) space. By shrinking R_0 if necessary, we can suppose that $R_0 < D_\kappa$.

Given $0 < r < R < R_0$, let $\Phi : B(x, R) \to B(x, r)$ be the map that sends y to the point y' on the (unique) geodesic $[x, y]$ which is at distance $\frac{r}{R} d(x, y)$ from x.

We claim that Φ is surjective. To see this, choose $y' \in B(x, r)$. The geodesic between x and y' can be extended to a local geodesic γ of length $\frac{R}{r} d(x, y') < R_0 < D_\kappa$. In a CAT($\kappa$) space, every local geodesic of length at most D_κ is actually a (global) geodesic, see ([9], Prop. II 1.4). It follows that γ is a geodesic. If y denotes its endpoint, $\Phi(y) = y'$. This proves surjectivity.

If $\kappa \leq 0$, the CAT(κ)-inequality implies that $d(\Phi(y_1), \Phi(y_2)) \leq \frac{r}{R} d(y_1, y_2)$. Therefore, the α-dimensional Hausdorff-measure of the image of Φ is at most

$\left(\frac{r}{R}\right)^\alpha \mathcal{H}^\alpha B(y,R)$. Since Φ is surjective, it follows that

$$\frac{\mathcal{H}^\alpha(B(y,r))}{b_\alpha r^\alpha} \leq \frac{\mathcal{H}^\alpha(B(y,R))}{b_\alpha R^\alpha},$$

from which the claimed existence of the limit follows.

If $\kappa > 0$, the argument is similar, although a bit more technical. We need the following lemma, taken from ([9], Lemma II 3.20).

Lemma 2.3. *For each $\kappa \in \mathbb{R}$, there exists a continuous function $C_\kappa : [0, D_\kappa) \to \mathbb{R}^{>0}$ such that $\lim_{R \to 0} C_\kappa(R) = 1$ and for all $x \in M^2_\kappa$, all $y_1, y_2 \in B(x,R)$ and all $0 \leq \epsilon \leq 1$*

$$C_\kappa(R)^{-1} \epsilon d(y_1,y_2) \leq d(\epsilon y_1, \epsilon y_2) \leq C_\kappa(R) \epsilon d(y_1,y_2),$$

where ϵy_1 denotes the point at distance $\epsilon d(x,y_1)$ from x on the geodesic between x and y_1.

We return to the proof of the proposition. The lemma and the CAT(κ) inequality imply that $d(\Phi(y_1),\Phi(y_2)) \leq C_\kappa(R) \frac{r}{R} d(y_1,y_2)$. Therefore,

$$\frac{\mathcal{H}^\alpha(B(x,r))}{b_\alpha r^\alpha} \leq C_\kappa(R)^\alpha \frac{\mathcal{H}^\alpha(B(x,R))}{b_\alpha R^\alpha}, \tag{1}$$

and the existence of the limit follows from $\lim_{R \to 0} C_\kappa(R) = 1$. □

Proposition 2.4. *Let X be a metric space with curvature bounded from above by κ which has the geodesic extension property. Then the function $\theta_\alpha : X \mapsto [0,\infty]$ is upper semi-continuous for all real numbers $\alpha \geq 0$.*

Proof: Fix some $x \in X$. It is sufficient to show that for all $\delta \in [0,\infty)$ and all sequences $y_1, y_2, \ldots$ converging to x, $\theta_\alpha(X, y_i) \geq \delta$ for all i implies $\theta_\alpha(X,x) \geq \delta$.

Let $\epsilon_i := d(x,y_i)$. By triangle inequality, $B(y_i, r - \epsilon_i) \subset B(x,r)$. We fix $r > 0$. For all sufficiently big i, we have $\epsilon_i < r$ and

$$\frac{\mathcal{H}^\alpha(B(x,r))}{b_\alpha r^\alpha} \geq \frac{\mathcal{H}^\alpha(B(y_i, r-\epsilon_i))}{b_\alpha r^\alpha} \geq \frac{\left(1 - \frac{\epsilon_i}{r}\right)^\alpha}{C_\kappa(r-\epsilon_i)^\alpha} \delta.$$

Letting i tend to infinity, we get

$$\frac{\mathcal{H}^\alpha(B(x,r))}{b_\alpha r^\alpha} \geq \frac{\delta}{C_\kappa(r)^\alpha}.$$

Taking the limit on both sides as $r \to 0$ and using $\lim_{r \to 0} C_\kappa(r) = 1$ we finally obtain $\theta_\alpha(X,x) \geq \delta$. □

The situation in spaces with curvature bounds from below is quite similar. The corresponding proposition is the following.

Proposition 2.5. *Let X be a space with curvature bounded below by $\kappa \in \mathbb{R}$. Then for each $x \in X$ and each $\alpha \geq 0$, the limit*

$$\theta_\alpha(X,x) := \lim_{r \to 0} \frac{\mathcal{H}^\alpha(B(x,r))}{b_\alpha r^\alpha} \in [0,+\infty]$$

exists. The function $x \mapsto \theta_\alpha(X,x)$ is lower semi-continuous.

Proof: Consider for $0 < r < R$ the map $\Phi : B(x,R) \to B(x,r)$ which sends a point y in $B(x,R)$ to the point at distance $\frac{r}{R}d(x,y)$ from x on some geodesic between x and y. Since geodesics in spaces with lower curvature bounds need not be unique, this map is in general neither unique nor continuous. But the Alexandrov inequality combined with Lemma 2.3 imply that

$$d(\Phi(y_1), \Phi(y_2)) \geq C_\kappa(R)^{-1}\frac{r}{R}d(y_1,y_2).$$

Passing to the α-dimensional Hausdorff measure yields

$$\frac{\mathcal{H}^\alpha B(x,r)}{b_\alpha r^\alpha} \geq C_\kappa(R)^{-\alpha}\frac{\mathcal{H}^\alpha B(x,R)}{b_\alpha R^\alpha}.$$

This inequality implies the existence of the limit for $r \to 0$, since $\lim_{R\to 0} C_\kappa(R) = 1$.

The upper semi-continuity is proved in a similar way as in the case of upper curvature bounds. It is enough to show that for all $\delta \in [0,\infty)$ and all sequences $y_1, y_2, \dots$ converging to x, $\theta_\alpha(X,y_i) \leq \delta$ for all i implies $\theta_\alpha(X,x) \leq \delta$.

Set $\epsilon_i := d(x,y_i)$. By triangle inequality, $B(x,r) \subset B(y_i, r+\epsilon_i)$ and therefore

$$\frac{\mathcal{H}^\alpha B(x,r)}{b_\alpha r^\alpha} \leq \frac{\mathcal{H}^\alpha B(y_i, r+\epsilon_i)}{b_\alpha\,(r+\epsilon_i)^\alpha}\left(1+\frac{\epsilon_i}{r}\right)^\alpha \leq C_\kappa(r+\epsilon_i)^\alpha\left(1+\frac{\epsilon_i}{r}\right)^\alpha\delta.$$

Letting i tend to ∞, and afterwards letting r tend to 0, we get $\theta_\alpha(X,x) \leq \delta$. □

2.3. **Quasi-geodesics.**

In spaces with lower curvature bounds, geodesics can not be extended in general, even if the space is a topological manifold. We therefore use quasi-geodesics as substitute for geodesics. The following lemmas and propositions are taken from [21] and [22].

Definition 2.6. *Let X be a metric space with curvature bounded from below by $\kappa \in \mathbb{R}$. A quasi-geodesic in X is a curve γ, which is parametrized by arclength, such that the following condition holds:*
Given points $x \in X, y_1 = \gamma(t_1), y_2 = \gamma(t_2), y_3 = \gamma(t_3), t_1 < t_2 < t_3, t_3 - t_1 \leq d(x,y_1) + d(x,y_3) < 2D_\kappa - (t_3 - t_1)$, choose a comparison triangle $\tilde{x}, \tilde{y}_1, \tilde{y}_3 \in M^2_\kappa$ such that $d(\tilde{x},\tilde{y}_1) = d(x,y_1), d(\tilde{x},\tilde{y}_3) = d(x,y_3), d(\tilde{y}_1,\tilde{y}_3) = t_3 - t_1$ and a comparison point $\tilde{y}_2$ on $[\tilde{y}_1,\tilde{y}_3]$ such that $d(\tilde{y}_2,\tilde{y}_1) = t_2 - t_1, d(\tilde{y}_2,\tilde{y}_3) = t_3 - t_2$. Then $d(x,y_2) \geq d(\tilde{x},\tilde{y}_2)$.

The assumption on the distances is made in order to have a triangle inequality for the comparison triangle $\tilde{x}, \tilde{y}_1, \tilde{y}_3$.

As a consequence from Alexandrov's lemma ([9], I 2.16), we get the following lemma, which will be central in the proof of Theorem 1.

Lemma 2.7. *Let γ be a quasi-geodesic in X. Let x, y_1, y_2, y_3 be points as in the definition of quasi-geodesics. Define $\tilde{\angle}_{y_2}(x,y_1)$ to be the angle at $\tilde{y}_2$ of a comparison triangle $\tilde{x}, \tilde{y}_1, \tilde{y}_2 \in M^2_\kappa$ such that $d(\tilde{x},\tilde{y}_1) = d(x,y_1), d(\tilde{x},\tilde{y}_2) = d(x,y_2), d(y_1,y_2) = t_2 - t_1$. Define $\tilde{\angle}_{y_2}(x,y_3)$ in a similar way. Then $\tilde{\angle}_{y_2}(x,y_1) + \tilde{\angle}_{y_2}(x,y_3) \leq \pi$.*

□

We now come to the important question when a quasi-geodesic can be extended. First note that each point in a metric space with lower curvature bound admits a tangent space $T_p^A X$ (in further applications, there will be another notion of tangent space and we write the superscript A to avoid confusion). It is defined as the cone over the completion of the set of germs of geodesics starting from this point, with the (Alexandrov) angle as distance. Two vectors $v_1, v_2 \in T_p^A X$ are said to be polar if $\angle(v_1, w) + \angle(v_2, w) \leq \pi$ for all $w \in T_p^A X$.

A right tangent vector at s of a Lipschitz curve γ is a limit point of $\{\frac{1}{t} \log_{\gamma(s)} \gamma(s+t), t > 0\}$. Left tangent vectors are similarly defined. Here $\log_x y$ is defined as the point at distance $d(x,y)$ from the origin on the image of a geodesic between x, y in $T_x^A X$. In general, log is a multi-valued function, since there may be several geodesics between x and y. In the same way, a curve can have different tangent vectors. But for quasi-geodesics, this phenomenon can not occur:

Lemma 2.8. *If γ is a quasi-geodesic, then left and right tangent vectors are unique and polar for all s.*

The two main properties of quasi-geodesics are the following.

Proposition 2.9. Gluing
Given two quasi-geodesics $\gamma_1 : (-\epsilon, 0] \to X, \gamma_2 : [0, \epsilon) \to X$ such that $\gamma_1(0) = \gamma_2(0)$ and such that the left tangent vector of γ_1 at 0 and the right tangent vector of γ_2 at 0 are polar. Then the concatenation of γ_1 and γ_2 is a quasi-geodesic.

Proposition 2.10. Existence
For any $x \in X, v \in T_p^A X, \|v\| = 1$, there exists a quasi-geodesic starting at x with initial right tangent vector v.

Corollary 2.11. *Let $\gamma : (-\epsilon, 0] \to X$ be a quasi-geodesic such that $x = \gamma(0)$ is a regular point, i.e. $T_x^A X$ is Euclidean space. Then γ can be extended beyond x.*

3. Subanalytic spaces, stratifications and generalized tangent spaces

3.1. **Subanalytic sets.** For the convenience of those readers who are not familiar with the theory of subanalytic sets, we will collect some facts which will be used in the sequel. Several good introductions into the theory of subanalytic sets, semialgebraic sets and o-minimal structures are available, see for instance [7],[15], [8].

Let M be a real analytic manifold. A subset $X \subset M$ is called semianalytic if each point $x \in M$ admits an open neighborhood $U \subset M$ such that $X \cap U = \cup_i \cap_j \{g_{ij} > 0\} \cap \{f_i = 0\}$ with (finitely many) analytic functions $g_{ij}, f_i : U \to \mathbb{R}$. The image of a semianalytic set $X \subset N$ under a proper analytic map $N \to M$, where N is a real analytic manifold, is called subanalytic subset of M. The space of subanalytic subsets of a real analytic manifold is closed under finite intersections, finite unions, taking complements and taking closures. Moreover, the image of a subanalytic set under a proper real analytic map is subanalytic.

We will need in this paper the fact that subanalytic sets have nice stratifications and that strata of codimension 1 behave very regularly.

3.2. **Whitney- and Verdier-stratifications.** Let M denote a real analytic manifold. A set $X \subset M$ is called stratified, if X can be written as disjoint, locally finite union of submanifolds of M, called strata, such that the boundary of each stratum is a union of strata.

For instance, a closed submanifold of M is a stratified set, with one single stratum. A submanifold with boundary has two strata: the smooth part and the boundary part.

Without imposing further conditions, not much can be said about stratified sets. The most common one is the following.

Definition 3.1. *A stratified subset X of a smooth N-dimensional manifold M satisfies Whitney's condition B at $x \in X$, if for all pairs S_1, S_2 of strata with $x \in S_1$, and one (and then each) smooth chart $\phi : U \to \mathbb{R}^N$ around x, the following condition is fulfilled:*

Let $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}$ be two sequences of points with $x_k \in S_1 \cap U, y_k \in S_2 \cap U, x_k \neq y_k, \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = x$ such that the lines $\phi(x_k)\phi(y_k)$ converge to a line L and such that the tangent spaces $d\phi_{y_k}(T_{y_k}S_2)$ converge to a limit space T. Then $L \subset T$.

The space X is said to satisfy condition B if this is the case for each $x \in X$.

Whitney's condition B has strong topological consequences, for instance it implies local topological triviality along strata (see [23], 3.9.3. for a proof of this result of Thom and Mather and further information). But for the study of metric properties, Verdier's condition gives more information.

Recall that the vector space distance between two vector spaces T and U of $\mathbb{R}^N$ is defined by $\delta(T, U) := \sup_{t\in T, \|t\|=1} d(t, U)$. In general, δ is not symmetric, but when restricted to subspaces of the same dimension, δ is a metric.

Definition 3.2. *A stratified subset X of a smooth manifold M of dimension N satisfies Verdier's condition at $x \in X$, if for each pair of strata S_1, S_2 with $x \in S_1$ and one (and then each) smooth chart $\phi : U \to \mathbb{R}^N$ around x, there exists an open neighborhood $U' \subset U$ of x and a constant $C_{Verdier} > 0$ such that for all $x' \in S_1 \cap U'$ and $y \in S_2 \cap U'$*

$$\delta(d\phi_{x'}(T_{x'}S_1), d\phi_y(T_yS_2)) \leq C_{Verdier}\|\phi(x') - \phi(y)\|. \qquad (v)$$

The set X satisfies Verdier's condition if this is the case for each $x \in X$.

In the subanalytic context, Verdier's condition is stronger than condition B: namely any subanalytic stratification of a subanalytic set which satisfies Verdier's condition also satisfies condition B. Moreover, each subanalytic set has such a Verdier stratification ([24]).

3.3. **Quasi-regular boundary points.**

Definition 3.3. *Let M be a smooth manifold of dimension N and let X be an n-dimensional C^1-submanifold of M. Then a boundary point $x \in \bar{X} \setminus X$ is called regular boundary point, if the germ of X at x is C^1-diffeomorphic to the germ at 0 of the set $\{x \in \mathbb{R}^N : x_n > 0, x_{n+1} = \cdots = x_N = 0\}$.*

The point x is called quasi-regular boundary point, if there exists a neighborhood U of x such that

a) *$U \cap \bar{X} \setminus X$ is a C^1 submanifold of dimension $n-1$,*
b) *$U \cap X$ has finitely many connected components $\Gamma_1, \ldots, \Gamma_k$, and*
c) *x is a regular boundary point of each $\Gamma_i, i = 1, \ldots, k$.*

The theorem of Pawłucki relates condition B and quasi-regularity.

Theorem 3.4. (Theorem of Pawłucki, [20])
Let $X \subset M$ be an n-dimensional Whitney B stratified subset of the smooth manifold M. Let $x \in X$ be contained in an $n-1$-dimensional stratum and let S be a highest (i.e. n-)dimensional stratum such that x is contained in the boundary of S. Then x is a quasi-regular boundary point of S.

In fact, Pawłucki's Theorem is a bit more general, but we will only need the above version of it.

3.4. **Generalized tangent spaces and local description at regular points.** Let X be a compact n-dimensional subanalytic subset of the real analytic N-dimensional manifold M. Choose a subanalytic stratification of X such that Verdier's condition is fulfilled. Then Whitney's condition B is fulfilled as well (see above). By Pawłucki's theorem, each point x of a stratum S of dimension $n-1$ is a quasi-regular boundary point for each n-dimensional stratum which has S on its boundary. This implies that there is some open neighborhood V of x in M such that $V \cap (X \setminus S)$ has a finite number of connected components and x is a regular boundary point of each of them.

Let Γ be one such component. The fact that x is a regular boundary point implies that (after shrinking V if necessary) there is an open set $V' \subset \mathbb{R}^N$ and a C^1-diffeomorphism $\phi = (\phi_1, \ldots, \phi_N) : V \to V'$ such that $\Gamma = \{p \in V : \phi_n(p) > 0, \phi_{n+1}(p) = \cdots = \phi_N(p) = 0\}$. Shrinking V again, we can suppose that $N := \{p \in V : \phi_{n+1}(p) = \cdots = \phi_N(p) = 0\}$ is a C^1-manifold.

The tangent space T_xN is then well-defined and is called *tangent space* of Γ and denoted by $T_x\Gamma$. It is an n-dimensional subspace of T_xM and inherits the natural Euclidean metric from T_xM. The tangent space T_xS is an $n-1$ dimensional subspace of T_xN (this follows from Whitney's condition B).

Let v_+ be the unit vector in $T_x\Gamma$ which is orthogonal to T_xS and such that $d\phi_n|_x(v_+) > 0$. We call v_+ the *inner pointing normal vector of S at x with respect to* Γ. We can write $T_x\Gamma$ as orthogonal sum: $T_x\Gamma = T_xS \oplus \mathbb{R}v_+$.

Definition 3.5. *The second fundamental form of S is the sum over the second fundamental forms II_{v_+} for all inner pointing normal vectors of S with respect to the components Γ of $V \cap (X \setminus S)$. Here, II_v denotes the second fundamental form of S in direction v.*

Let us now suppose that $M = \mathbb{R}^N$ is Euclidean space with the canonical metric. Using the implicit function theorem (after shrinking V if necessary), we get that the orthogonal projection $P = P_{T_xN}|_N$ from N to T_xN is a C^1-diffeomorphism. The image of Γ under this projection is clearly given by $\{p \in T_xN : \phi_n \circ P^{-1}(p) > 0\}$. We note that $\phi_n \circ P^{-1}$ is a C^1-function. Using the implicit function theorem, we get a C^1-function $g : T_pS \to \mathbb{R}$ such that

$$P(\Gamma) = \{p = (p_1, p_2) \in T_xN = T_xS \oplus \mathbb{R} : g(p_1) > p_2\}.$$

We claim that g is even a smooth function. To see this, note that the restriction of P to S is smooth, since S is a smooth submanifold. Therefore the image of S under P is a smooth submanifold of T_xN, which equals the set $\{(p_1, p_2) \in T_pN : g(p_1) = p_2\}$.

Let us summarize what will be needed in the sequel from this section.

Corollary 3.6. *Let X be an n-dimensional compact subanalytic subset of $\mathbb{R}^N$, the latter being equipped with its standard metric. Then there exists a stratification of X satisfying Verdier's condition. For each point $x \in X$ in a stratum S of dimension $n-1$, there exists an open ball $B = B(x, r)$ around x, such that the following conditions are fulfilled.*

a) *The intersection $B \cap (X \setminus S)$ is a finite union of connected C^1 manifolds with boundary $B \cap S$.*
b) *For each such manifold Γ there exists a unique limit tangent space at x, which will be denoted by $T_x\Gamma$. This means that $\lim_{y \to x, y \in \Gamma} T_y\Gamma \to T_x\Gamma$.*
c) *We have an orthogonal splitting $T_x\Gamma = T_xS \oplus \mathbb{R}v_+$ where v_+ is the inner pointing normal vector of S with respect to Γ.*
d) *The projection from Γ to $T_x\Gamma$ is given by the set $\{(p_1, p_2) \in T_x\Gamma \cap P_T(B) : g(p_1) > p_2\}$ with a smooth function $g : T_xS \cap P_T(B) \to \mathbb{R}$.*
e) *Verdier's condition (v) is fulfilled with some constant $C_{Verdier} > 0$ for all $x' \in S \cap B, y \in \Gamma \cap B$.*
f) *Whitney's condition is fulfilled with some constant $C_{Whitney} > 0$ in the following form: for all $y \in \Gamma \cap B$, $\|P_{T_x\Gamma}(x-y)\| \geq C_{Whitney}\|x-y\|$.*

4. Asymptotic behavior of angles

Given a point x in an $n-1$-dimensional stratum S of a Verdier stratified n-dimensional compact subanalytic subset $X \subset \mathbb{R}^N$, we choose a ball B around x as in Corollary 3.6. Then $B \cap (X \setminus S)$ is a finite union of connected C^1 manifolds with boundary $B \cap S$. Let Γ be one such manifold.

Consider a rectifiable curve $\gamma : [0, t_{max}) \to \Gamma \cap S$ with $t_{max} > 0, \gamma(0) = x$. We suppose that γ is parameterized by arclength and that the inner pointing normal vector v_+ of S with respect to Γ is the unique tangent vector of γ at 0, i.e. that $\gamma(t) = x + tv_+ + r(t)$, where $r : [0, t_{max}) \to \mathbb{R}^N$ is a function such that $\lim_{t\to 0} \frac{\|r(t)\|}{t} = 0$. In later applications, γ will be a geodesic or a quasi-geodesic.

With S being smooth, there is a real $s_{max} > 0$ such that the exponential map of S at x is defined for all vectors of norm at most $s_{max} > 0$. Let $v \in T_xS$ be a unit tangent vector of S at x. Define the curve β by $\beta(s) := \exp_x sv, |s| \leq s_{max}$. Note that $II_{v_+}(v, v) = \langle \beta''(0), v_+\rangle$.

For fixed s, t we consider a comparison triangle in M^2_κ (if it exists) $\Delta(\tilde{x}, \tilde{\beta}(s), \tilde{\gamma}(t))$ with side lengths $d(\tilde{x}, \tilde{\beta}(s)) = d(x, \beta(s)), d(\tilde{x}, \tilde{\gamma}(t)) = t, d(\tilde{\beta}(s), \tilde{\gamma}(t)) = d(\beta(s), \gamma(t))$. This is not a comparison triangle in the usual sense, since t can be bigger than $d(x, \gamma(t))$. We do not know for the moment if such a comparison triangle exists, since triangle inequality could be violated. But in the case that we will consider, triangle inequality will be trivial.

We denote by $\angle_x^{(\kappa)}(\beta(s), \gamma(t))$ the angle at $\tilde{x}$ of the comparison triangle $\Delta(\tilde{x}, \tilde{\beta}(s), \tilde{\gamma}(t))$.

The following proposition relates the asymptotic behavior of this angle with the second fundamental form of S in direction v_+.

Proposition 4.1. *Given $\delta > 0$, there exist $C_1 = C_1(\delta, x, \beta, \gamma) > 0$ such that for all $C > C_1$*

$$\limsup_{s\to 0} \frac{1}{s} \left|\cos \angle_x^{(\kappa)}(\beta(s), \gamma(Cs^2)) + \cos \angle_x^{(\kappa)}(\beta(-s), \gamma(Cs^2)) - sII_{v_+}(v, v)\right| \leq \delta.$$

It is easy to see that both angles in the lemma tend to $\frac{\pi}{2}$, so the sum of the cosines tends to 0. The proposition gives an estimate on the deviation from 0. This deviation is determined up to a small error term by the second fundamental form. Once we have further information about these angles, for instance in the presence of upper or lower bounds of the curvature in Alexandrov's sense, we can use this lemma to obtain bounds on the second fundamental form.

Proof:

Step 1:

Let P_T denote the orthogonal projection from $\mathbb{R}^N$ to the limit tangent space $T_x\Gamma$. By Corollary 3.6, $P_T\Gamma$ is given by the points $\{(p_1, p_2) \in T_x\Gamma \cap P_T(B) : g(p_1) < p_2\}$.

With x, β fixed we can choose positive constants C_2, C_3, C_4, C_5 such that for all s of sufficiently small absolute value we have the following bounds:

a) $s \geq d(x, \beta(s)) \geq \|x - \beta(s)\| \geq s - C_2s^3$
b) $\left\|\beta(s) - x - s\beta'(0) - \frac{s^2}{2}\beta''(0)\right\| \leq C_3s^3$
c) $|\langle P_T\beta(s) - x, v_+\rangle| \leq C_4s^2$

d) $\| \operatorname{grad} g(y) \| \le C_5 s$ for all $y \in T_x S$ with $\|y - x\| \le s$.

This follows from a Taylor expansion of β and the fact that $\beta'(0) \perp v_+$.

We set

$$C_1 = C_1(\delta, x, \beta, \gamma)$$
$$:= \sup \left\{ \frac{1}{2}\langle \beta''(0), v_+ \rangle, 4C_5 + 2C_4, \frac{2C_3 + 2C_2 + \frac{1}{4}\langle \beta''(0), v_+\rangle^2 + \frac{4C_{Verdier}^2}{C_{Whitney}^2}}{\delta} \right\}.$$

Fix some $C > C_1$.

Step 2: To find a lower bound for $d(\beta(s), \gamma(Cs^2))$, we will use the Euclidean distance. To find an upper bound, we will project Γ to the tangent space $T_x\Gamma$, join the images of $\beta(s)$ and $\gamma(Cs^2)$ by a straight line and lift this line to a curve in $\Gamma \cup S$. To estimate its length, we will need Verdier's condition. Let us come to the details.

In $T_x\Gamma$, we consider the line $u \mapsto (1-u)P_T\gamma(Cs^2) + uP_T\beta(s), 0 \le u \le 1$. We claim that for sufficiently small $s > 0$, it is contained in $(P_T\Gamma \cup P_T S) \cap P_T(B)$.

To prove the claim, consider the smooth function $h(u) := (1-u)p_2 + uq_2 - g((1-u)p_1 + uq_1)$, where $P_T\gamma(Cs^2) = (p_1, p_2)$ and $P_T\beta(s) = (q_1, q_2)$ with respect to the splitting $T_x\Gamma = T_x S \oplus \mathbb{R}v_+$. Note that $h(1) = q_2 - g(q_1) = 0$, since $P_T\beta(s) \in P_T S$.

From the asymptotic of γ and the inequalities above, we infer that for sufficiently small $s > 0$, $\|p_1 - x\| \le s, p_2 \ge \frac{1}{2}Cs^2, \|q_1 - x\| \le s, |q_2| \le C_4 s^2$.

If there is a zero of h in $(0, 1)$, there would be a zero $\xi \in (0, 1)$ of the derivative of h. Then

$$\frac{1}{2}Cs^2 - C_4 s^2 \le |q_2 - p_2| \le \| \operatorname{grad} g((1-\xi)p_1 + \xi q_1)\| \cdot |q_1 - p_1| \le 2C_5 s^2,$$

which contradicts the choice of C. Since $h(0) > 0$, we must have $h > 0$ on $(0, 1)$ which proves the claim.

Define the smooth curve $\eta_s : [0, 1] \to X$ as the lift of the line to $\Gamma \cup S$, i.e. by $\eta_s(u) \in \Gamma \cup S; P_T\eta_s(u) = (1-u)P_T\gamma(Cs^2) + uP_T\beta(s)$. Clearly η_s is a curve between $\beta(s)$ and $\gamma(Cs^2)$, therefore $d(\beta(s), \gamma(Cs^2)) \le length(\eta_s)$.

Step 3: Let $Q_{u,s} := (P_T)|_{T_{\eta_s(u)}\Gamma} : T_{\eta_s(u)}\Gamma \to T_x\Gamma$ denote the orthogonal projection and $Q_{u,s}^{-1} : T_x\Gamma \to T_{\eta_s(u)}\Gamma$ its inverse.

The distance from x to the line between $P_T\gamma(Cs^2)$ and $P_T\beta(s)$ is clearly bounded by $\max\{s, Cs^2\}$ and tends to 0 for $s \to 0$. From Corollary 3.6, f), it follows that the distance from x to η_s also tends to 0 for $s \to 0$.

Using the convergence of the tangent spaces we obtain some strictly positive continuous function $\phi_{cont} : (0, \infty) \to \mathbb{R}$ with $\lim_{s\to 0} \phi_{cont}(s) = 0$ such that for all $u \in (0, 1)$ and all sufficiently small $s > 0$ we have $\|Id - Q_{u,s}^{-1}\| \le \phi_{cont}(s)$.

Step 4: The derivative of η_s at a point $0 < u < 1$ is given by the orthogonal lift of the vector $w := P_T\beta(s) - P_T\gamma(Cs^2) \in T_x\Gamma$ to $T_{\eta_s(u)}\Gamma$. According to the orthogonal splitting $T_x\Gamma = T_xS \oplus \mathbb{R}v_+$, we decompose $w = w_1 + w_2$ with $w_1 \in T_xS$ and $w_2 \parallel v_+$. Then $Q_{u,s}^{-1}w = Q_{u,s}^{-1}w_1 + Q_{u,s}^{-1}w_2$.

The reason we do this is that for the lift of w_1 we have a strong estimate coming from Verdier's condition. For the lift of w_2, there is only a weaker estimate induced by the continuity of the tangent map. On the other hand, $\|w_2\|$ is very short compared to $\|w_1\|$, so that we finally will get a good upper bound for the length of the lift.

Let us first consider the *projection* $P_{\eta_s(u)}w_1$ of w_1 to $T_{\eta_s(u)}\Gamma$. From Verdier's condition, we get

$$\|w_1 - P_{\eta_s(u)}w_1\| \leq C_{Verdier}\|x - \eta_s(u)\| \cdot \|w_1\|.$$

With $\bar{w} := w_1 - P_T P_{\eta_s(u)}w_1$ we get

$$\|\bar{w}\| \leq \|w_1 - P_{\eta_s(u)}w_1\| \leq C_{Verdier}\|x - \eta_s(u)\| \cdot \|w_1\|,$$

since P_T is a projection and $P_Tw_1 = w_1$.

Therefore

$$\begin{aligned}
\|Q_{u,s}^{-1}w_1 - w_1\| &= \|Q_{u,s}^{-1}\bar{w} + P_{\eta_s(u)}w_1 - w_1\| \\
&\leq \|Q_{u,s}^{-1}\bar{w}\| + C_{Verdier}\|x - \eta_s(u)\| \cdot \|w_1\| \\
&\leq (1 + \phi_{cont}(s))\|\bar{w}\| + C_{Verdier}\|x - \eta_s(u)\| \cdot \|w_1\| \\
&\leq (2 + \phi_{cont}(s))C_{Verdier}\|x - \eta_s(u)\| \cdot \|w_1\|.
\end{aligned}$$

The length of w_1 is bounded by $\|w\|$ and, supposing that $Cs^2 \leq s$, i.e. $s \leq C^{-1}$,

$$\|x - \eta_s(u)\| \leq C_{Whitney}^{-1}\|x - P_T\eta_s(u)\| \leq C_{Whitney}^{-1}s.$$

On the other hand, by Step 3,

$$\|w_2 - Q_{u,s}^{-1}w_2\| \leq \phi_{cont}(s)\|w_2\|.$$

The norm of w_2 is bounded by

$$\begin{aligned}
\|w_2\| &= |\langle w, v_+\rangle| \\
&= \left|\left\langle P_T\gamma(Cs^2) - P_Tx, v_+\right\rangle - \langle P_T\beta(s) - x, v_+\rangle\right| \\
&\leq \|\gamma(Cs^2) - x\| + C_4s^2 \\
&\leq Cs^2 + C_4s^2.
\end{aligned}$$

It follows that

$$\|w - Q_{u,s}^{-1}w\| \leq C_{Verdier}(2 + \phi_{cont}(s))C_{Whitney}^{-1}s\|w\| + \phi_{cont}(s)(C + C_4)s^2. \quad (2)$$

Step 5: We need the asymptotic development for $\|w\|^2$. Complete v_+ and v to an orthogonal base $v_+, v, v_1, \ldots, v_{n-2}$ of $T_x\Gamma$. Then $\|w\|^2 = \langle w, v_+\rangle^2 + \langle w, v\rangle^2 + \sum_{i=1}^{n-2}\langle w, v_i\rangle^2$.

$$\begin{aligned}
&\left|\langle w, v_+\rangle - \frac{s^2}{2}\langle \beta''(0), v_+\rangle + Cs^2 + \langle r(Cs^2), v_+\rangle\right| \\
&\leq \underbrace{\left|\langle P_T\beta(s) - x, v_+\rangle - \frac{s^2}{2}\langle \beta''(0), v_+\rangle\right|}_{\leq C_3 s^3} + \underbrace{\left|\langle x - P_T\gamma(Cs^2), v_+\rangle + Cs^2 + \langle r(Cs^2), v_+\rangle\right|}_{=0} \\
&\leq C_3 s^3. \quad (3)
\end{aligned}$$

Since β is a geodesic on S, $\beta''(0) \perp T_x S$ and therefore

$$\left|\langle w, v\rangle - s + \langle r(Cs^2), v\rangle\right| \leq |\langle P_T\beta(s) - x - sv, v\rangle| \leq C_3 s^3.$$

For $i = 1, 2, \ldots, n-2$ we obtain

$$\left|\langle w, v_i\rangle + \langle r(Cs^2), v_i\rangle\right| = |\langle P_T\beta(s) - x, v_i\rangle| \leq C_3 s^3,$$

i.e. $\langle w, v_i\rangle^2 \leq o(s^4)$.

From the preceding estimates, we get for $s \to 0$,

$$\|w\|^2 \leq s^2 - 2s\langle r(Cs^2), v\rangle + s^4\left(C^2 + \frac{1}{4}\langle \beta''(0), v_+\rangle^2 - C\langle \beta''(0), v_+\rangle + 2C_3\right) + o(s^4). \quad (4)$$

Similarly,

$$\|w\|^2 \geq s^2 - 2s\langle r(Cs^2), v\rangle + s^4\left(C^2 + \frac{1}{4}\langle \beta''(0), v_+\rangle^2 - C\langle \beta''(0), v_+\rangle - 2C_3\right) + o(s^4).$$

As a first consequence we have $\|w\| \geq \frac{1}{2}s$ provided s is sufficiently small. Replacing this into equation 2, we get with $C_6 = C_6(s, C) := C_{Verdier}(2 + \phi_{cont}(s))C_{Whitney}^{-1} + 2\phi_{cont}(s)(C + C_4)$

$$\|w - Q_{u,s}^{-1}w\| \leq C_6 s\|w\|.$$

By construction, we have the orthogonal decomposition $Q_{u,s}^{-1}w = w + (Q_{u,s}^{-1}w - w)$ from which we deduce

$$\|Q_{u,s}^{-1}w\|^2 \leq (1 + C_6^2 s^2)\|w\|^2.$$

It follows that the length of the curve η_s, and hence the distance between $\beta(s)$ and $\gamma(Cs^2)$ is bounded by

$$\begin{aligned}
d(\beta(s), \gamma(Cs^2))^2 &\leq (1 + C_6^2 s^2)\|w\|^2 \\
&\leq s^2 - 2s\langle r(Cs^2), v\rangle + \\
&\quad + s^4\left(C^2 + \frac{1}{4}\langle \beta''(0), v_+\rangle^2 - C\langle \beta''(0), v_+\rangle + 2C_3 + C_6^2\right) + o(s^4).
\end{aligned}$$

On the other hand, since P_T does not increase distances,

$$
\begin{aligned}
d(\beta(s),\gamma(Cs^2))^2 &\geq \|w\|^2 \\
&\geq s^2 - 2s\langle r(Cs^2), v\rangle + \\
&\quad + s^4\left(C^2 + \frac{1}{4}\langle \beta''(0), v_+\rangle^2 - C\langle \beta''(0), v_+\rangle - 2C_3\right) + o(s^4).
\end{aligned}
$$

Step 6: Replacing this into the formula for the cosine in the case $\kappa = 0$ yields

$$\cos\angle_x^{(0)}(\beta(s),\gamma(Cs^2)) \leq \frac{s}{2}\left(\langle \beta''(0), v_+\rangle + 2\frac{C_3}{C}\right) + \frac{1}{Cs^2}\langle r(Cs^2), v\rangle + o(s).$$

The angle $\cos\angle_x^{(0)}(\beta(-s),\gamma(Cs^2))$ can be bounded in a similar way, taking care that now we have to use $-v$ instead of v. The result is

$$\cos\angle_x^{(0)}(\beta(-s),\gamma(Cs^2)) \leq \frac{s}{2}\left(\langle \beta''(0), v_+\rangle + 2\frac{C_3}{C}\right) - \frac{1}{Cs^2}\langle r(Cs^2), v\rangle + o(s).$$

Taking the sum of both inequalities and passing to the lim sup, we obtain

$$
\begin{aligned}
&\limsup_{s\to 0}\frac{1}{s}\left(\cos\angle_x^{(0)}(\beta(s),\gamma(Cs^2)) + \cos\angle_x^{(0)}(\beta(-s),\gamma(Cs^2))\right) \\
&\qquad\qquad \leq \langle \beta''(0), v_+\rangle + \frac{2C_3}{C} \leq \langle \beta''(0), v_+\rangle + \delta.
\end{aligned}
$$

Using the upper bound for $d(\beta(s),\gamma(Cs^2))$ and proceeding as before (taking into account that $\lim_{s\to 0} C_6(C,s) = 2C_{Verdier}C_{Whitney}^{-1}$), we also get

$$
\begin{aligned}
&\liminf_{s\to 0}\frac{1}{s}\left(\cos\angle_x^{(0)}(\beta(s),\gamma(Cs^2)) + \cos\angle_x^{(0)}(\beta(-s),\gamma(Cs^2))\right) \\
&\geq \langle \beta''(0), v_+\rangle - \frac{2C_3+2C_2}{C} - \frac{1}{4C}\langle \beta''(0), v_+\rangle^2 - \frac{2C_{Verdier}^2}{C_{Whitney}^2 C} \geq \langle \beta''(0), v_+\rangle - \delta.
\end{aligned}
$$

This finishes the proof of Proposition 4.1 in the case $\kappa = 0$. The general case then follows easily from the following lemma.

Lemma 4.2. *Let $\angle^{(\kappa)}(a,b,c)$ denote the angle (in face of c) of a triangle in M_κ^2 with side lengths $a,b,c, a \geq b > 0$. Then*

$$\lim_{a,b\to 0; a\geq b}\frac{1}{a}\left(\cos\angle^{(0)}(a,b,c) - \cos\angle^{(\kappa)}(a,b,c)\right) = 0.$$

Proof of the Lemma: The case $\kappa = 0$ is trivial, so let us assume $\kappa \neq 0$. We set $\cos_\kappa(x) := \cos(\sqrt{\kappa}x), \sin_\kappa(x) := \sin(\sqrt{\kappa}x)$ (if $\kappa < 0$ this is still defined as a complex-valued function).

The law of cosines in M_κ^2 reads ([9], I 2.13)

$$\cos\angle^{(\kappa)}(a,b,c) = \frac{\cos_\kappa c - \cos_\kappa a \cos_\kappa b}{\sin_\kappa a \sin_\kappa b}.$$

Replacing the Taylor development $\cos_\kappa(x) = 1 - \kappa\frac{x^2}{2} + \sum_{i\geq 4} a_i x^i, \sin_\kappa(x) = \sqrt{\kappa}x - \sum_{i\geq 3} b_i x^i$ in this formula we easily get the statement of the lemma, and hence the general case of the proposition. □

5. Proof of the comparison theorem in the case of an upper curvature bound

We recall that M is a real analytic manifold with a Riemannian metric g and $X \subset M$ an n-dimensional compact connected subanalytic subset. We suppose that X is a topological manifold. The metric g induces on X an inner metric, denoted by d.

Using a local real analytic isometric embedding of M in an Euclidean space, we can suppose that $M = \mathbb{R}^N$.

5.1. Proof of $K \leq \kappa \implies ric \leq (n-1)\kappa$.

We suppose that X has curvature bounded from above by κ in the sense of Alexandrov (see Section 2). Since the density function is upper semi-continuous on such a space, and since $\theta(X,x) = 1$ for a dense set of points, namely for all points in highest-dimensional strata, we get $\theta(X,x) \geq 1$ for all $x \in X$.

Choose a stratification of X according to Corollary 3.6.

Since the condition on the upper curvature bound is a local one, the sectional curvature of each highest dimensional stratum is bounded from above by κ. It follows that the Ricci curvature of such a stratum is bounded from above by $(n-1)\kappa$.

Now consider a stratum S of codimension 1 and $x \in S$. From the local description given in Corollary 3.6, we get some small $r > 0$, such that $B(x,r) \cap (X \setminus S)$ is a finite union of connected C^1-manifolds with boundary $B(x,r) \cap S$. The fact that X is a topological manifold implies that there are exactly two such manifolds, denoted by Γ_1, Γ_2. Let v_+^i denote the inner pointing normal vector of S with respect to $\Gamma_i, i = 1,2$.

Suppose there is a geodesic $\gamma : [0, t_{max}) \to S \cup \Gamma_1$ with $\gamma(0) = x$ and such that v_+^1 is the unique (right) tangent vector of γ at 0. Metric spaces with upper curvature bound which are topological manifolds have the geodesic extension property. We deduce that γ can be extended to a geodesic $\gamma : (t_{min}, t_{max}) \to X$, $t_{min} < 0 < t_{max}$. It is intuitively clear that the negative part of γ will be in Γ_2 and that the left tangent vector of γ at 0 is v_+^2. In fact, this is a consequence of the law of reflection of [2]. It also follows from it that there is a dense set of points x through which passes such a geodesic.

There is a real number $s_{max} > 0$ such that the exponential map of S at x is defined for all vectors of norm at most $s_{max} > 0$. Let $v \in T_xS$ be a unit tangent vector of S at x. Define the curve β by $\beta(s) := \exp_x sv, |s| \leq s_{max}$.

Let $\delta > 0$ be given. Then, by Proposition 4.1, there exists a $C_1 > 0$ such that for all $C > C_1$

$$\limsup_{s \to 0} \frac{1}{s} \left| \cos \angle_x^{(\kappa)}(\beta(s), \gamma(Cs^2)) + \cos \angle_x^{(\kappa)}(\beta(-s), \gamma(Cs^2)) - sII_{v_+^1}(v,v) \right| \leq \delta.$$

In this inequality, the angles denote comparison angles in the classical sense, since the geodesic property of γ implies that $d(x, \gamma(t)) = t$ for all $t \geq 0$.

Applying the same proposition to Γ_2, there is a $C_2 > 0$ such that for all $C > C_2$ the same inequality is satisfied with $\gamma(Cs^2)$ replaced by $\gamma(-Cs^2)$ and v_+^1 replaced by v_+^2. We choose $C > \max\{C_1, C_2\}$.

The upper curvature bound and the fact that γ is a geodesic imply that for $s > 0$, the sum of the angles $\angle_x^{(\kappa)}(\beta(s), \gamma(Cs^2))$ and $\angle_x^{(\kappa)}(\beta(s), \gamma(-Cs^2))$ is at least π. From the monotony of the cosine function, we get

$$\cos \angle_x^{(\kappa)}(\beta(s), \gamma(Cs^2)) + \cos \angle_x^{(\kappa)}(\beta(s), \gamma(-Cs^2)) \leq 0,$$

and similarly

$$\cos \angle_x^{(\kappa)}(\beta(-s), \gamma(Cs^2)) + \cos \angle_x^{(\kappa)}(\beta(-s), \gamma(-Cs^2)) \leq 0.$$

Replacing this into the inequality above implies that $II(v, v) \leq \delta$. Since $\delta > 0$ and v were arbitrary, we finally have $II \leq 0$.

From Corollary 3.6 a) we deduce that the vectors v_+ and v_- depend continuously on the base point x, therefore II depends continuously on x. We conclude that $II \leq 0$ on all of S. □

5.2. **Proof of $K \leq \kappa$ implies $E \leq \binom{n-1}{2}\kappa$.** On highest dimensional strata, the implication is easy and classical. As above, the density is at least 1 in each point. On strata S of codimension 1, we get by the above reasoning that $II \leq 0$ and this implies that $\operatorname{tr} II g|_S - II \leq 0$. □

5.3. **Proof of $ric \leq (n-1)\kappa \implies s \leq n(n-1)\kappa$.**
Now suppose that X has Ricci curvature bounded from above by $(n-1)\kappa$. On highest dimensional strata, this implies that the classical Ricci curvature is bounded from above by $(n-1)\kappa$ and then the classical scalar curvature is bounded from above by $n(n-1)\kappa$. On strata of codimension 1, the upper Ricci-curvature bound means that the second fundamental form is non-positive, therefore the same is true for mean curvature of such a stratum. Finally, by definition, the density at each point is at least 1. □

5.4. **Proof of $E \leq \binom{n-1}{2}\kappa \implies s \leq n(n-1)\kappa$ (if $\dim X \geq 3$).**

Just take the trace of E on highest-dimensional strata and the trace of $\operatorname{tr} II g|_S - II$ on strata S of codimension 1. The argument breaks down in dimension 2, since then $E = 0$ and $\operatorname{tr} II g|_S - II = 0$. □

6. Proof of the comparison theorem in the case of a lower curvature bound

6.1. **Proof of $K \geq \kappa \implies ric \geq (n-1)\kappa$.**
Suppose that X has sectional curvature bounded from below by $\kappa \in \mathbb{R}$ in the sense of Alexandrov (see 1.1). We want to show that its Ricci curvature is bounded from below by $(n-1)\kappa$.

Choose a stratification according to Corollary 3.6. On highest dimensional strata, Alexandrov's condition is equivalent to the classical sectional curvature being bounded from below by κ. And this implies that the Ricci tensor is bounded from below by $(n-1)\kappa g$.

Every point in a highest-dimensional stratum has density 1. From the lower semi-continuity of the density function in Alexandrov spaces of curvature bounded below, we get that the density is at most 1 at each point.

Again, as in the case of an upper curvature bound, the difficulty lies in the strata of codimension 1. Let $x \in S$ be a point in a stratum of codimension 1. Locally, we again have the description of Corollary 3.6. Suppose that y is a point in Γ_1 such that x is the nearest point to y on S. This will happen for a dense set of points x, as was indicated in the previous section.

Connect x and y by a geodesic γ. If we can extend γ beyond x, then we can proceed as in the case of an upper curvature bound and use Proposition 4.1 to show that the second fundamental form of S at the point x is non-negative. There is, however, the problem that in general γ can not be extended. On the other hand, what we really need to accomplish the proof along the same lines as in the case of an upper curvature bound is not the fact that γ is a geodesic, but a certain behavior of angles which also holds true for quasigeodesics, and quasigeodesics can be extended through regular points.

There are two technical problems related to this approach. First we shall show that a point $x \in S$ is regular in the sense of Alexandrov, i.e. its (Alexandrov) tangent space is Euclidean space. Secondly, we need an asymptotic formula of the form $\gamma(t) = x + tv_+ + o(t)$ for a quasigeodesic with (Alexandrov) initial vector v_+. This will be achieved in the next two lemmas.

Lemma 6.1. *Let X be a compact subanalytic set. Suppose that X is an n-dimensional topological manifold and that X has curvature bounded from below by $\kappa \in \mathbb{R}$ in the sense of Alexandrov. Let $x \in S$ be a point in an $n-1$-dimensional stratum of a Verdier stratification of X. Then x is regular in the sense of Alexandrov, i.e. the Alexandrov tangent space is isometric to $\mathbb{R}^n$.*

Proof: Denote the Alexandrov tangent space of X at x by $T_x^A X$.

Choose an orthonormal base $v_1, \ldots, v_{n-1}$ of T_xS. For $i = 1, \ldots, n-1$, choose a sequence $y_1^i, y_2^i, \ldots$ of points in S with limit x such that $\frac{y_j^i - x}{\|y_j^i - x\|} \to v_i$. Denote by γ_i one limit of the sequence $\{\log_x y_j^i, j = 1, 2, \ldots\}$. Using $-v_i$ instead of v_i, we obtain a limit γ_i'. For the moment, we do not know if these limits are unique, but it will turn out from this lemma that they actually are unique.

By Corollary 3.6, a small neighborhood of x consists of the union of S and two C^1-manifolds Γ_1 and Γ_2, with corresponding normal vectors v_+^1 and v_+^2. Choose a sequence of points $y_1, y_2, \ldots$ in Γ_1 converging to x such that

$\frac{y_i - x}{\|y_i - x\|} \to v_+^1$. Let $\gamma_n \in T_x^A X$ be one limit of the sequence $\{\log_x y_i \in T_x^A X\}$. Using v_+^2 instead of v_+^1, we obtain a point $\gamma_n' \in T_x^A X$.

We claim that

- $\angle(\gamma_i, \gamma_j) \geq \frac{\pi}{2}, \angle(\gamma_i', \gamma_j) \geq \frac{\pi}{2}, \angle(\gamma_i', \gamma_j') \geq \frac{\pi}{2}$ if $i \neq j$.
- $\angle(\gamma_i, \gamma_i') = \pi$ for all $i = 1, \ldots, n$.

We just sketch the proof of the claim (see [1] for more details). First, Whitney's condition implies that the (inner) distance from x to some point $y \in X$ which is near x is nearly the same as the Euclidean distance. This means that both differ by a factor which tends to 1 if y tends to x. The length of the third side of a triangle xy_1y_2 can be bounded from below by the Euclidean distance between y_1, y_2. Passing to the limit, one obtains the inequality above. The equality is proved in a similar way for $i = 1, \ldots, n-1$. For $i = n$ one uses the fact that $y \in \Gamma_1$ sufficiently close to x will lie in a tubular neighborhood of S and that each geodesic between two points $y_1 \in \Gamma_1, y_2 \in \Gamma_2$ near x has to pass through S. We skip the (easy) details.

Now we can finish the proof of the lemma. By the second statement, we get that the union of $\mathbb{R}_+\gamma_1$ and $\mathbb{R}_+\gamma_1'$ is a geodesic line in $T_x^A X$. From the splitting theorem ([11], 10.5.1.), we get $T_x^A X = \mathbb{R} \times Y$ for some non-negatively curved space Y. The first condition implies that $\gamma_i, \gamma_i' \in Y$ for $i = 2, \ldots, n$. Then we can apply the splitting theorem again to split off another $\mathbb{R}$-factor. Continuing this way, we finally get $T_x^A X = \mathbb{R}^n$.

The inequalities above show furthermore that the points $\gamma_i, \gamma_i', i = 1, \ldots, n$ do not depend on the particular choice of sequences $y_1, y_2, \ldots$ and are uniquely defined. □

Lemma 6.2. *Let $\gamma_n = \lim_{i \to \infty} \log_x y_i$, where $y_1, y_2, \ldots$ is a sequence of points of Γ_1 converging to x such that $\frac{y_i - x}{\|y_i - x\|} \to v_+^1$. Let γ be a quasi-geodesic starting at x with initial (Alexandrov) tangent vector γ_n. Then $\lim_{t \to 0} \frac{\gamma(t) - x}{t} = v_+^1$.*

Proof: Set $t_i := \|y_i - x\| \to 0$. By definition of γ_n and the uniqueness of the tangent vector for quasigeodesics, $\frac{\log_x y_i}{t_i} \to \gamma_n$ and $\frac{\log_x \gamma(t_i)}{t_i} = \gamma_n$. Therefore

$$\lim_{i \to \infty} d\left(\frac{1}{t_i} \log_x \gamma(t_i), \frac{1}{t_i} \log_x y_i\right) = 0.$$

The log-function is distance non-decreasing, which implies that

$$\frac{d(\gamma(t_i), y_i)}{t_i} \leq \frac{d(\log_x \gamma(t_i), \log_x y_i)}{t_i} \to 0.$$

Finally we get

$$\left\| \frac{\gamma(t_i) - x - t_i v_+^1}{t_i} \right\| \leq \frac{d(\gamma(t_i), y_i)}{t_i} + \frac{\|y_i - x - t_i v_+^1\|}{t_i} \to 0.$$

The same reasoning works for any sequence of points $y_1, y_2, \ldots$ with $y_i \to x, y_i \in \Gamma_1, \frac{y_i - x}{\|y_i - x\|} \to v_+^1$, since γ_n does not depend on the particular choice of such a sequence (see proof of Lemma 6.1). It follows that

$$\lim_{t \to 0} \frac{\gamma(t) - x - tv_+^1}{t} = 0.$$

□

We can now finish the proof of Theorem 1 in the case of lower curvature bounds.

There is a dense set of points $x \in S$ such that there is a point $y \in \Gamma_2$ with the property that x is the nearest point on S to y. By continuity of II, it suffices to show that II is non-negative on this set. Let $x \in S, y \in \Gamma_2$ be a pair of such points.

Consider a geodesic between y and x. By simple variation arguments (which are made explicit in [2]), $\gamma(t) = x + |t|v_+^2 + o(|t|), t < 0$. The last lemma implies that the tangent vector in the Alexandrov sense is given by $\gamma_n' \in T_x^A X = \mathbb{R}^n$. We can extend γ as a quasi-geodesic beyond x, as x is regular (see Lemma 6.1 and Corollary 2.11). The right tangent vector of γ at 0 is given by γ_n. The previous lemma applied to the positive part of γ yields the asymptotic behavior $\gamma(t) = x + tv_+^1 + o(t), t > 0$.

Let $v \in T_x S$ be a unit tangent vector. Define the curve β by setting for $|s| < s_{max}$ $\beta(s) := \exp_x sv$, where $\exp_x$ is the exponential map of S and s_{max} the injectivity radius of S at x.

Let $\delta > 0$ be given. Then, by Proposition 4.1, there exists a $C_1 > 0$ such that for all $C > C_1$

$$\limsup_{s \to 0} \frac{1}{s} \left| \cos \angle_x^{(\kappa)}(\beta(s), \gamma(Cs^2)) + \cos \angle_x^{(\kappa)}(\beta(-s), \gamma(Cs^2)) - sII_{v_+^1}(v,v) \right| \leq \delta.$$

Applying the same proposition to Γ_2, there is a $C_2 > 0$ such that for all $C > C_2$ the same inequality is satisfied with $\gamma(Cs^2)$ replaced by $\gamma(-Cs^2)$ and v_+^1 replaced by v_+^2. We choose $C > \max\{C_1, C_2\}$.

The fact that γ is a quasi-geodesic implies that for $s > 0$, the sum of the angles $\angle_x^{(\kappa)}(\beta(s), \gamma(Cs^2))$ and $\angle_x^{(\kappa)}(\beta(s), \gamma(-Cs^2))$ is at most π (see Lemma 2.7). From the monotony of the cosine function, we get

$$\cos \angle_x^{(\kappa)}(\beta(s), \gamma(Cs^2)) + \cos \angle_x^{(\kappa)}(\beta(s), \gamma(-Cs^2)) \geq 0$$

and similarly

$$\cos \angle_x^{(\kappa)}(\beta(-s), \gamma(Cs^2)) + \cos \angle_x^{(\kappa)}(\beta(-s), \gamma(-Cs^2)) \geq 0.$$

Replacing this into the inequality above implies that $II(v,v) \geq -\delta$. Since v and $\delta > 0$ were arbitrary, we finally have $II \geq 0$.

The proof of the first part of Theorem 1 in the case of a lower curvature bound is now complete.

6.2. **Proof of $K \geq \kappa$ implies $E \geq \binom{n-1}{2}\kappa$.** Again, the result is trivial for strata of codimension 0. By the arguments above, each point in X has density at most 1. On strata S of codimension 1, we have $II \geq 0$ and this implies that $\operatorname{tr} IIg|_S - II \geq 0$. □

6.3. **Proof of $ric \geq (n-1)\kappa \implies s \geq n(n-1)\kappa$.**
Suppose X has Ricci curvature bounded from below by $(n-1)\kappa$. This implies that on highest dimensional strata, the classical Ricci curvature is bounded from below by $(n-1)\kappa$ and therefore the scalar curvature is bounded from below by $n(n-1)\kappa$. On strata of codimension 1, the lower Ricci curvature bound implies that the second fundamental form and therefore also the mean curvature is non-negative. Finally, the density at each point is at most 1 by definition of the Ricci curvature bound. We conclude that if the Ricci curvature of X is at least $(n-1)\kappa$, then its scalar curvature is at least $n(n-1)\kappa$. □

6.4. **Proof of $E \geq \binom{n-1}{2} \implies s \geq n(n-1)\kappa$ (if $\dim X \geq 3$).**

Just take the trace of E on highest-dimensional strata and the trace of $\operatorname{tr} IIg|_S - II$ on strata of codimension 1. As in the case of upper curvature bounds, the argument does not work if $\dim X = 2$, since then $E = \operatorname{tr} IIg|_S - II = 0$. □

This finishes the proof of Theorem 1 in the case of lower curvature bounds. □.

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